An Approach to Stabilize Linear Systems with State and Input Delay

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Abstract

A sliding mode control scheme has been designed to ensure the asymptotic stability of a linear system with delay in both the input and state variables. A key step is the use of a transformation that eliminates the input delay. The resulting state-delay system is controlled through a sliding mode approach where the equivalent control term uses state feedback and the discontinuous control term involves the sign and value of the switching function. It is shown that the sliding mode is reached in finite time. Stability conditions are derived using a Lyapunov technique combined with a Razumikhin approach to bound the difference between the original and the transformed states. The result is a set of sufficient conditions for asymptotic stability. The theoretical developments are illustrated via an example.

1 Introduction

The presence of delay in a dynamic system often poses a significant challenge to ensure closed loop stability [1]. Sliding mode control (SMC), a technique known for its robustness with respect to perturbations and system uncertainties, has been used to stabilize systems with time delays; however, most of the literature focuses on systems with either state delay [2]-[5] or with input delay [6] [7]. Little work has been done regarding systems with simultaneous state and input delays [8] [9].

A novelty of this paper is an approach proposed to address the problem of stabilizing a linear system featuring both state and input delay. First a state transformation is used to map the original system into an input-delay free form where only state delays are present. Then, a sliding mode control law is proposed to stabilize the system and to ensure that the state trajectories reach the sliding surface in finite time.

In Section 2 the problem is presented along with the transformation that eliminates the input delay. The design of the control law is discussed in Section 3. In section 4 the control law is shown to drive the system states to the sliding surface in finite time, and this result is followed in Section 5 by the derivation of sufficient conditions for the asymptotic stability of the original closed-loop system. The paper includes an example in Section 6, and presents concluding remarks in Section 7.

2 Problem Formulation

Consider the system

$$\dot{x}(t) = Ax(t) + A_d x(t-h) + Bu(t) + B_d u(t-h) \tag{1}$$

$$x(\tau) = \Phi(\tau),\ \ \tau \in [-h, 0]$$

$$u(\tau) = \Psi(\tau),\ \ \tau \in [-h, 0]$$

where, $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the control input, and A, A_d, B, and B_d are matrices of appropriate dimensions. The system delay h is considered to be constant, $\Phi(\tau)$ is an initial-state function, and $\Psi(\tau)$ is an initial-input function. The notation $|\cdot|$ is used to indicate, dependent on the scalar or vector nature of the argument, an absolute value of a scalar quantity or a vector norm, and $\|\cdot\|$ is used to indicate an induced matrix norm.

2.1 System Transformation

The following state transformation is introduced as suggested in [11] to map (1) into an input-delay free system:

$$z(t) = x(t) + \int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau \tag{2}$$

Differentiating equation (2) to obtain

$$\dot{z}(t) = \dot{x}(t) + A\int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau + e^{-Ah} B_d u(t) - B_d u(t-h)$$

and then substituting for $\dot{x}(t)$ from (1) gives

$$\dot{z}(t) = Az(t) + A_d x(t-h) + \bar{B}u(t)$$

where $\bar{B} = B + e^{-Ah} B_d$. In this work it is assumed that the pair $(A, \bar{B})$ is controllable. Let

$$v(t) := \int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau \tag{3}$$

so that the transformed system becomes

$$\dot{z}(t) = Az(t) + A_d z(t-h) + \bar{B}u(t) + \hat{A}_d v(t-h) \quad (4)$$

where $\hat{A}_d = -A_d$. Note that from (2), $v(t) = z(t) - x(t)$ is interpreted as the difference between the original system $x(t)$ and the transformed system $z(t)$. A feedback matrix F is introduced such that $\bar{A} = A - \bar{B}F$ is Hurwitz [12]. Treating the last term in (4) as an internal disturbance and defining $f(t, v(t-h)) := \hat{A_d} v(t-h)$, the system equation (4) can be rewritten as

$$\begin{aligned} \dot{z}(t) &= (\bar{A} + \bar{B}F)z(t) + A_d z(t-h) + \bar{B}u(t) \\ &\quad + f(t, v(t-h)) \end{aligned} \quad (5)$$

which is free from input delay.

3 Control Law Design

The sliding surface is defined by a scalar switching function $s(t) \in \Re$ of the form

$$s(t) = Cz(t) - \int_0^t [C\bar{A}z(\tau) + CA_d z(\tau - h)]\, d\tau \quad (6)$$

where C is a design matrix chosen such that $C\bar{B}$ is nonsingular. The structure of the control law is given by

$$u(t) = u_e(t) + u_d(t) \quad (7)$$

where $u_e(t)$ is the equivalent part and $u_d(t)$ is the discontinuous part of the control law. The equivalent control is obtained by setting to zero the derivative of equation (6) with respect to time, and then solving for $u(t)$ to yield

$$\dot{s}(t) = C\dot{z}(t) - C\bar{A}z(t) - CA_d z(t-h) = 0$$

Following the standard approach in SMC, the state derivative $\dot{z}(t)$ in the above equation is taken from (5) after ignoring the disturbance term $f(t, v(t-h))$. This gives the identity

$$C(\bar{A} + \bar{B}F)z(t) + CA_d z(t-h) + C\bar{B}u(t) - C\bar{A}z(t)$$

$$-CA_d z(t-h) = 0$$

which reduces to

$$C\bar{B}Fz(t) + C\bar{B}u(t) = 0$$

The solution to the above identity is $u(t) = u_e(t)$; hence, after recognizing that $C\bar{B}$ is invertible it is possible to conclude that the equivalent control law sought is

$$u_e(t) = -Fz(t) \quad (8)$$

The discontinuous control law proposed is

$$u_d(t) = -(C\bar{B})^{-1}[ks(t) + \rho(t)\ sgn(s(t))] \quad (9)$$

where

$$\rho(t) = \|C\|\ \|\hat{A_d}\|\ |v(t-h)| + \zeta \quad (10)$$

and where $k > 0$ and $\zeta > 0$ are design parameters, and $v(t-h) = z(t-h) - x(t-h)$.

4 Existence of a Sliding Mode

Theorem 1 *The time-delay system (5) with control law (7)-(10) reaches the sliding manifold within a finite time t_s, where*

$$t_s = \frac{1}{k}\ \ln\left(1 + \frac{k\ |s(0)|}{\zeta}\right) \quad (11)$$

Proof: Select $V(t) = \frac{1}{2}s(t)^2$ as a candidate scalar Lyapunov function. Then,

$$\begin{aligned} \dot{V}(t) &= s(t)\dot{s}(t) = s(t)(C\dot{z}(t) - C\bar{A}z(t) - CA_d z(t-h)) \\ &= s(t)\{C(\bar{A} + \bar{B}F)z(t) + CA_d z(t-h) \\ &\quad + C\bar{B}\ [-Fz(t) - (C\bar{B})^{-1}(ks(t) + \rho(t)\ sgn(s))\ ] \\ &\quad + Cf(t, v(t-h)) - C\bar{A}z(t) - CA_d z(t-h)\ \} \\ &= s(t)(-ks(t) - \rho(t)\ sgn(s)) + Cf(t, v(t-h)) \end{aligned}$$

Now, since $|Cf(t, v(t-h))| \le \|C\|\ \|\hat{A_d}\|\ |v(t-h)|$, after invoking (10) it follows that

$$\dot{V}(t) \le -k\ |s(t)|^2 - \zeta\ |s(t)| \quad (12)$$

Therefore, $\dot{V}(t) < 0$ for all $k > 0$ and $\zeta > 0$, and it can be concluded that the system trajectories attain sliding mode in finite time. An estimate for the upper bound of the reaching time t_s can be obtained by integrating the differential equation $\dot{V}(t) = -k|s(t)|^2 - \zeta|s(t)|$, where $|s(t)| = \sqrt{2V(t)}$, under the initial condition $V(0) = \frac{1}{2}s(0)^2$. The result (11) is obtained after a simple transformation of variables and straightforward algebraic manipulations. ■

5 System Stability

At sliding mode the control law (7) reduces to $u(t) = u_e(t)$. Then, from (8) it follows that the dynamic system (5) is given by the expression

$$\dot{z}(t) = \bar{A}z(t) + A_d z(t-h) + f(t, v(t-h)) \quad (13)$$

The developments in the suite make use of the inequality

$$|v(t)| \le \eta(h)\ |z(t)| \quad (14)$$

where

$$\eta(h) = h \max_{0\leq\theta\leq h} \|e^{-A\theta}\| \, \|B_d\| \, \|F\|\alpha \quad (15)$$

and where $\alpha > 1$ is a Razumikhin constant describing the evolution of $|z(t)|$, *i.e.*, $|z(\theta)| \leq \alpha \, |z(t)|$, $\theta \in [t-h, t]$. The bound (14) follows from applying successive bounding operations to the right-hand side of (3) and introducing the Razumikhin parameter.

Theorem 2 *The time delay system (5) with control law (7)-(10) is asymptotically stable at sliding mode if there exist positive-definite matrices* $P \in \Re^{n\times n}$, $R \in \Re^{n\times n}$, *and* $Q \in \Re^{n\times n}$ *such that*

$$\lambda_{min}(R) > \lambda_{max}(Q) \quad (16)$$

and

$$\lambda_{min}(Q)(\lambda_{min}(R) - \lambda_{max}(Q)) > (1+\eta(h))^2 \, \|PA_d\|^2 \quad (17)$$

where P and R are solutions to the Lyapunov equation

$$P\bar{A} + \bar{A}^T P = -R \quad (18)$$

Proof: Consider a Lyapunov functional of the form

$$V(t) = z^T(t)Pz(t) + \int_{t-h}^{t} z^T(\tau)Qz(\tau)d\tau \quad (19)$$

The time derivative of $V(t)$ with respect to time is given by

$$\dot{V}(t) = 2z^T(t)P\dot{z}(t) + (z^T(t)Qz(t) - z^T(t-h)Qz(t-h))$$

Substituting the expression for $\dot{z}(t)$ given in (13) yields

$$\begin{aligned}\dot{V}(t) \quad = \quad & 2z^T(t)P\bar{A}z(t) + 2z^T(t)PA_d z(t-h) \\ & +2z^T(t)Pf(t, v(t-h)) + z^T(t)Qz(t) \\ & -z^T(t-h)Qz(t-h)\end{aligned} \quad (20)$$

and then using (18) and bounding the right-hand side of (20) yields

$$\begin{aligned}\dot{V}(t) \quad \leq \quad & -\lambda_{min}(R)|z(t)|^2 + \lambda_{max}(Q)|z(t)|^2 \\ & +2\|PA_d\||z(t)| \, |z(t-h)| \\ & +2\|P\hat{A_d}\| \, |z(t)| \, |v(t-h)| - \lambda_{min}(Q)|z(t-h)|^2\end{aligned} \quad (21)$$

Invoking the bound (14) and rearranging terms, inequality (21) can be written in the form

$$\dot{V}(t) \leq \begin{bmatrix} z(t)^T \\ z(t-h)^T \end{bmatrix}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z(t) \\ z(t-h) \end{bmatrix} \quad (22)$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{bmatrix} \lambda_{max}(Q) - \lambda_{min}(R) & (1+\eta(h)) \, \|PA_d\| \\ (1+\eta(h)) \, \|PA_d\| & -\lambda_{min}(Q) \end{bmatrix} \quad (23)$$

It suffices to show that the matrix (23) is negative-definite. This is ensured if and only if conditions (16) and (17) are satisfied. ■

Theorem 2 can be reformulated to show explicitly the constraint on the size of the delay parameter imposed by design choices, such as the adopted Lyapunov matrices R and Q.

Corollary 1 *The time-delay system (5) with control law (7)-(10) is asymptotically stable in sliding mode for time-delay values satisfying*

$$h \max_{0\leq\theta\leq h} \|e^{-A\theta}\| \leq \left(\frac{1}{2}\frac{\lambda_{min}(R)}{\|PA_d\|} - 1\right)\frac{1}{\|B_d\| \, \|F\| \, \alpha} \quad (24)$$

where P and R are positive-definite solutions to the Lyapunov equation (18) and satisfy the inequality

$$\lambda_{min}(R) > \max(\lambda_{max}(Q), 2\|PA_d\|) \quad (25)$$

Proof: The proof consists of deriving conditions that ensure the existence of a feasible solution to (16) and (17). The proof also recognizes that $\lambda_{max}(Q)$ represents the maximum eigenvalue to the Lyapunov functional (19), hence $\lambda_{min}(Q) \leq \lambda_{max}(Q)$. Using the latter inequality along with the constraint imposed on $\lambda_{min}(Q)$ by (17), it follows that

$$\frac{(1+\eta(h))^2\|PA_d\|^2}{\lambda_{min}(R) - \lambda_{max}(Q)} < \lambda_{min}(Q) \leq \lambda_{max}(Q) \quad (26)$$

A solution $\lambda_{min}(Q)$ to (26) exists only if

$$\frac{(1+\eta(h))^2\|PA_d\|^2}{\lambda_{min}(R) - \lambda_{max}(Q)} < \lambda_{max}(Q) \quad (27)$$

which, using the fact that (16) requires that $\lambda_{min}(R) - \lambda_{max}(Q) > 0$, is equivalent to

$$\lambda_{max}(Q)^2 - \lambda_{min}(R)\lambda_{max}(Q) + (1+\eta(h))^2\|PA_d\|^2 < 0 \quad (28)$$

The analysis of the above inequality reduces to investigating the boundary defined by the equality

$$\lambda_{max}(Q)^2 - \lambda_{min}(R)\lambda_{max}(Q) + (1+\eta(h))^2\|PA_d\|^2 = 0 \quad (29)$$

which can be readily solved to yield

$$\lambda_{max}(Q) = \frac{\lambda_{min}(R)}{2} \pm \frac{1}{2}\sqrt{\lambda_{min}(R)^2 - 4(1+\eta(h))^2\|PA_d\|^2}$$

Given that only real solutions are meaningful, it follows that the discriminant must be nonnegative, *i.e.,*

$$\lambda_{min}(R) \geq 2(1+\eta(h))\|PA_d\| \quad (30)$$

The presence of the factor $(1+\eta(h)) > 1$ implies that a feasible solution to (30) exists only if

$$\lambda_{min}(R) > 2\|PA_d\| \quad (31)$$

Furthermore, from (30) it follows that the set of feasible solutions is given by the equivalent inequality

$$\eta(h) \leq \frac{1}{2}\frac{\lambda_{min}(R)}{\|PA_d\|} - 1$$

which establishes condition (24) of the corollary after using (15) and suitably rearranging the factors in the inequality. Moreover, since $\lambda_{min}(R)$ must simultaneously satisfy condition (16) and constraint (31), it follows that it must satisfy condition (25) of the corollary. ■

It remains to show the asymptotic stability of system (1), as addressed in the following theorem.

Theorem 3 *The time-delay system (1) with state $x(t)$ is asymptotically stable if the transformed system (5) reaches the sliding manifold and is asymptotically stable on the manifold.*

Proof: If $z(t)$ reaches the sliding surface, then the control law reduces to $u(t) = -Fz(t)$, and (2) can be rearranged in the form

$$x(t) = z(t) + \int_{t-h}^{t} e^{A(t-h-\tau)} B_d F z(\tau) d\tau \tag{32}$$

Now when (5) is asymptotically stable, it follows that $z(t) \to 0$, then $x(t) \to 0$ in (32), hence completing the proof. ■

6 Example

Consider the time-delay system (1) with $h = 0.8$, with an initial-input function $\Psi(\tau) = 0$ for $\tau \in [-h, 0)$, and an initial-state function $\Phi(\tau) = [-1 \quad 2]^T$ for $\tau \in [-h, 0]$ so that the initial-state vector is $x(0) = [-1 \quad 2]^T$, and the following system parameters:

$$A = \begin{bmatrix} -1 & 0 \\ 0.2 & 0.3 \end{bmatrix}, \; A_d = \begin{bmatrix} 0.01 & -0.04 \\ 0.02 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \; B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The control design considered is based on the following matrices associated with the Lyapunov equation (18) and the Lyapunov functional (19):

$$R = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}, \; Q = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

$$P = \begin{bmatrix} 4.5435 & 5.7537 \\ 5.7537 & 48.4578 \end{bmatrix}$$

Using the feedback matrix $F = [-0.0166 \quad 0.2827]$ the eigenvalues of $\bar{A}$ are placed at $\{-0.4, \; -0.45\}$. The controller parameters are $k = 30$ and $\zeta = 3$. The switching function's initial value is $s(0) = 3$, and its design matrix is chosen as $C = [0.9 \quad 0.85]$. Selecting $\alpha = 2.3391$, calculating the norms of $\|B_d\| = 1$, $\|F\| = 0.2832$, and evaluating $\max_{0\leq\theta\leq h} \|e^{-A\theta}\| = 2.2381$, then equation (15) gives $\eta(h) = 1.1858$. It is now straightforward to verify that conditions (16) and (17) of Theorem 2 are satisfied. First, condition (16) is met given that $\lambda_{min}(R) = 6$ is greater than $\lambda_{max}(Q) = 3$. Also, condition (17) is met since $\lambda_{min}(Q)(\lambda_{min}(R) - \lambda_{max}(Q)) = 9$ is greater than $(1+\eta(h))^2 \; \|PA_d\|^2 = 5.4770$. It follows then that Theorem 2 ensures the asymptotic stability of the closed loop system. The conditions of Corollary 1 are also satisfied since this corollary is equivalent to Theorem 2. In fact, $h \max_{0\leq\theta\leq h} \|e^{-A\theta}\| = 1.7905$ is less than $(\frac{1}{2}\frac{\lambda_{min}(R)}{\|PA_d\|} - 1)\frac{1}{\|B_d\| \; \|F\| \; \alpha} = 2.7208$, and $\lambda_{min}(R) = 6$ is greater than $\max(\lambda_{max}(Q), 2\|PA_d\|) = 3$.

Figure 1 shows the results of a simulation study. Figure 1(a) depicts the asymptotic stability of the transformed system (5) with state variable $z(t)$. The state trajectories for the original system (1) with state variable $x(t)$ are shown in Figure 1(b). Equation (11) yields $t_s = 0.1145$, a value that is consistent with the time at which $s(t)$ becomes identically zero in Figure 1(d), given that at that instant $z(t)$ has reached the sliding manifold. It is apparent that the states $x(t)$ develop asymptotic behavior after a time $t \approx t_s + h = 0.9145$, which is a consequence of the fact that the original system has an input delay whereas the transformed system is free of input delay. Figure 1(c) shows the control action $u(t)$ rising quickly from its initial value, and reaching the value of zero at approximately the same time that $z(t)$ reaches the sliding manifold. Figure 1(c) also shows that the control scheme suffers from a chattering effect, as is to be expected from the presence of the *signum* function in the discontinuous control law (9).

The chattering of the signal can be alleviated by introducing the approximation

$$sgn(s) \approx \frac{s}{|s| + \varepsilon}$$

Figure 2 shows that a value of $\varepsilon = 0.001$ effectively makes the chattering disappear (see Figure 2(c)), while the state trajectories $z(t)$, $x(t)$, and the switching function $s(t)$ remain virtually unchanged.

7 Conclusions

A sliding mode controller has been designed to stabilize a linear system with state and input delay. The controller is shown to successfully drive the system states to the sliding surface in finite time. Sufficient stability conditions are derived using Lyapunov techniques.

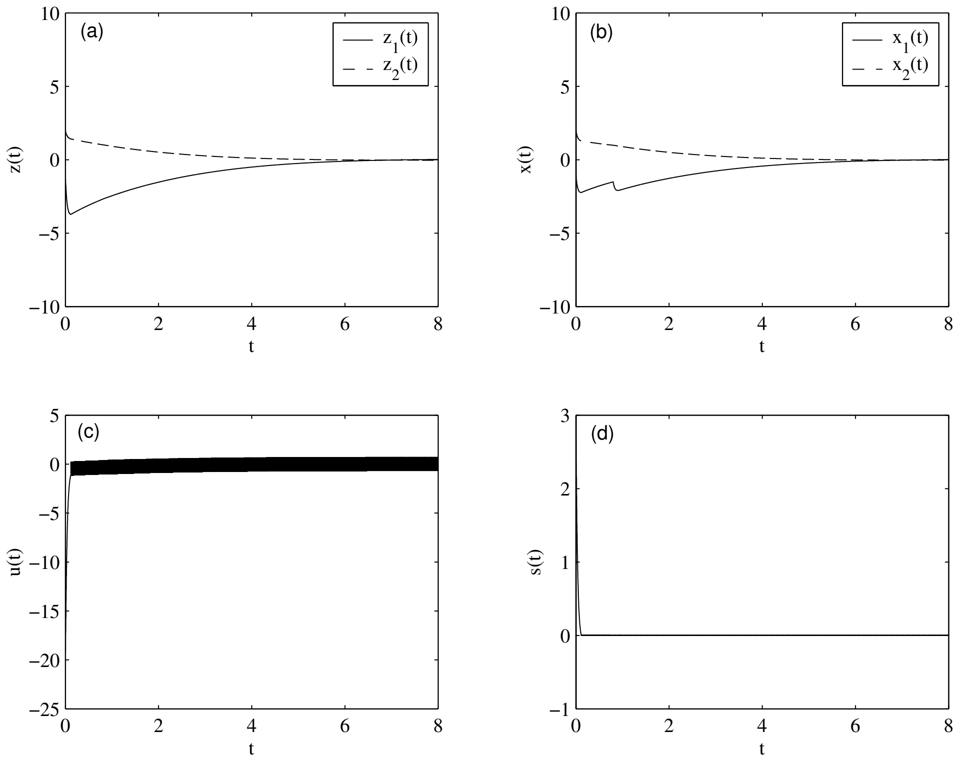

Figure 1: Trajectories for the states of the transformed system (a), the states of the original system (b), the control law (c), and the switching function (d).

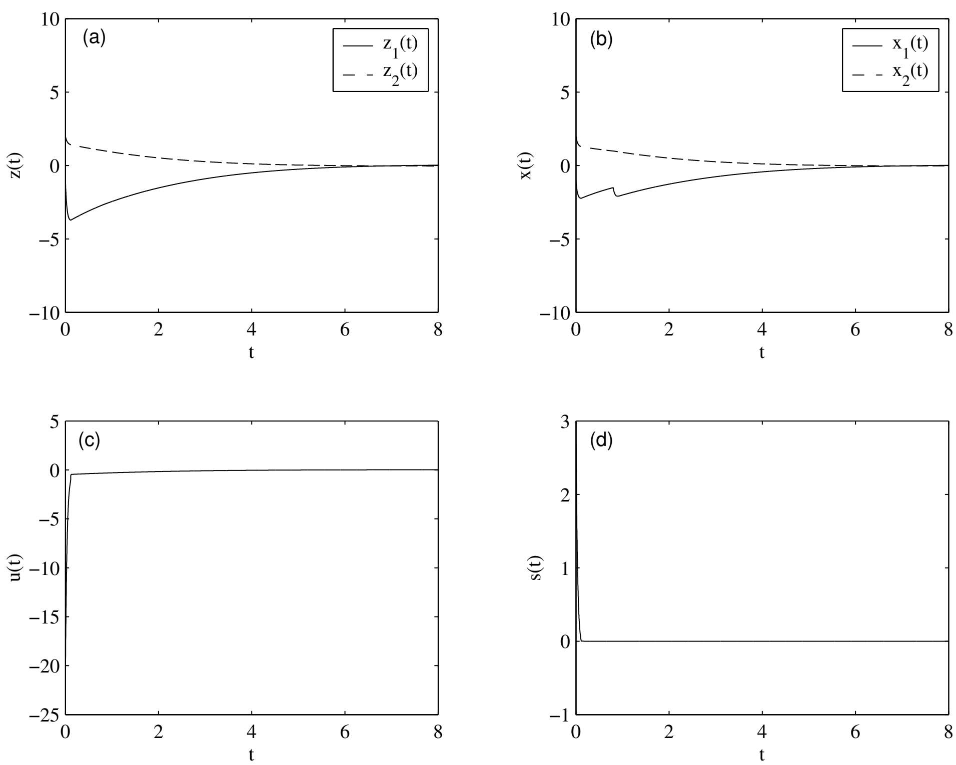

Figure 2: Trajectories for the states of the transformed system (a), the states of the original system (b), the control law (c), and the switching function with approximation to the signum function (d).

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