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Robust $\mathcal{H}$-Infinity Stabilization for Interval Plants

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Abstract

Given a system with real-parametric uncertainties, this paper presents an approach to determine the maximum stabilizable uncertainties and to design a controller which can stabilize the family of uncertain plants. The paper explores the critical direction theory to identify non-conservative weighting functions which are then utilized in an iterative H_∞ design process.

1 Introduction

The robust synthesis of a controller that can stabilize a plant with parametric perturbations is a largely unsolved problem for which only few results are available [2] [3] [4] [5] [7]. One of the first attempts was made by Bhattacharyya *et al.* [2], where parametric uncertainties were overbounded by unstructured uncertainties, and *static* constant overbounding uncertainty was used as the weighting function in an H_∞ method to design a robust controller. That approach is very simple and effective in making use of the well-developed H_∞ synthesis knowledge. Unfortunately, the overbounding operation may introduce conservatism; in fact it is pointed out in [2] that a significant deficiency of control theory is the lack of non-conservative synthesis methods to achieve robustness under parameter uncertainties. Many researchers [3] [5] [6] attack the synthesis problem using fixed-structure controllers such as PID, or other low-order controllers, but this approach suffers from inherent shortcomings by restricting attention to only fixed-structure controllers.

The recently proposed critical direction theory provides necessary and sufficient stability conditions for uncertain systems [8]. Motivated by this theory, Ji *et al.* [7] revisited the *static* weighting strategy introduced in [2] to reduce the conservatism resulting from the overbounding weighting function and obtained an increase in maximum stabilizable uncertainty of about 46% over previous results reported in [2] for examples widely referred to in the literature. However, as pointed out in [7], some conservatism might still exist by restricting attention to only static weighting. This paper first utilizes a *dynamic* weighting strategy to obtain an even larger stabilizable uncertainty.

2 Problem Formulation

This paper studies problems of the kind posed in [2] [4] [5] [7], where for a given interval system $p(s)$, as large a stabilizable uncertainty as possible is required to be found together with a robust controller that can stabilize the closed-loop system of Fig. 1.

The family of uncertain plants $p(s)$ is defined about its nominal system $p_0(s)$, namely,

$$p(s) = p_0(s) + \delta(s). \tag{1}$$

The system $p(s)$ is characterized by a set of real parameters which appear in the coefficients of the numerator and denominator polynomials. Each parameter in turn is an element of a real interval of width 2ϵ centered about the nominal value of the corresponding parameter for the nominal plant $p_0(s)$. Therefore, the uncertainty description for the problem is a real hyperbox $\Delta(\epsilon)$ each side of which has a length 2ϵ. The value ϵ is a non-negative scalar that represents the uncertainty level. In general, the nominal system $p_0(s)$ is neither open-loop stable nor unity-negative-feedback stable. Therefore, a *stabilizing* controller $c(s)$ needs to be designed so that the closed-loop system in Fig. 1 is robustly stable, *i.e.*, stable over all uncertain plants $p(s)$. To solve this robustness synthesis problem, an upper bound ϵ_u needs to be found for which the family of interval plant $p(s)$ satisfies the following condition:

(A) The set of allowable perturbations $\Delta(\epsilon)$ is such that systems $p(s)$ and $p_0(s)$ have the same number of open-loop unstable poles.

Whenever a controller $c(s)$ is designed, the stability of the closed-loop system of $g(s) = p(s)c(s)$ of Fig. 1 needs to be checked; and a stability margin needs to be determined. For robustness analysis of the system $g(s)$, the following conditions should be satisfied:

(B1) The nominal system $g_0(s) := p_0(s)c(s)$ is stable under unity-negative-feedback.
(B2) The family of uncertain systems $g(s)$ and its nominal system $g_0(s)$ have the same number of open-loop

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unstable poles.

In summary, the problem addressed in this paper is to determine the parametric stability margin ϵ_{max}, which, as defined in [2], is the maximum value ϵ for which the entire family of interval plants $p(s)$ is stabilizable. Note that ϵ_{max} can be no greater than ϵ_u. Another problem is to synthesize a stabilizing controller $c_{rc}(s)$ that can robustly stabilize the family of plants for any $\epsilon \le \epsilon_{max}$.

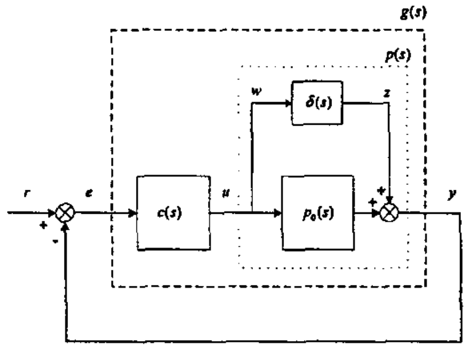

Figure 1: Feedback loop involving the uncertain system $p(s)$ and a controller $c(s)$.

3 Preliminaries

3.1 H_∞ Synthesis with Constant Overbounding Uncertainty Weighting

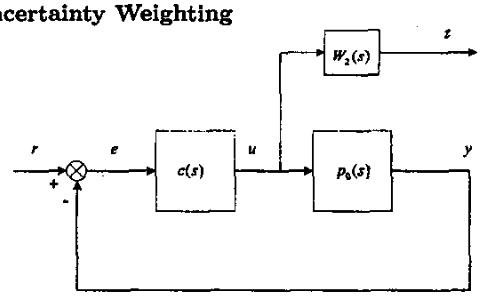

Figure 2: Robustness synthesis for parametric uncertain systems recast into the mixed-sensitivity framework.

Following the well-known small gain theory, the robustness synthesis problem formulated in Section 2 can be mapped into the mixed-sensitivity H_∞ synthesis framework, as shown in Fig. 2, where the controller design procedure involves finding the minimizing solution to

$$\gamma = \min_{stabilizing\ c(s)} \|W_2(s)R(s)\|_\infty, \qquad (2)$$

where $R(s) := T_{ur}(s) = \frac{c(s)}{1+c(s)p_0(s)}$ is the controller sensitivity transfer function, and $W_2(s)$ must be selected to represent the plant perturbations in as exact a manner as possible [9]. The choice of the weighting function $W_2(s)$ is critical in representing the *effective* part of the uncertainty $\delta(s)$. In general, the more closely the weighting function approximates the sufficient and necessary stability conditions, the less conservatism is introduced in solving for a robust controller. If $\gamma < 1$, the minimizing solution $c(s)$ is a robustly stabilizing controller. Note that the problem given in (2) is actually a simple controller sensitivity problem for which other sensitivity weighting functions are zero, *i.e.*, $W_1(s) = W_3(s) = 0$.

Bhattacharyya *et al.* [2] introduced the constant overbounding weighting method, namely, replacing $W_2(s)$ with a constant worst-case circular bound. Let $\rho_m(\omega)$ denote the maximum perturbation radius (MPR) of the system $p(j\omega)$ at each frequency ω, namely

$$\rho_m(\omega) := \max_{\epsilon \in \Delta(\epsilon)} |\delta(j\omega, \epsilon)| = \max_{\epsilon \in \Delta(\epsilon)} |p(j\omega, \epsilon) - p_0(j\omega)|. \qquad (3)$$

The static weight used in [2] is denoted here as $\rho_m := \max_{\omega} \rho_m(\omega)$, which is defined as the constant overbounding uncertainty bound, *i.e.*, the largest value of $\rho_m(\omega)$ over all frequencies.

Since ρ_m encompasses all sizes of uncertain plants measured from the nominal plant over a frequency band of interest, it seems reasonable, at least as a first step, to use ρ_m as the weighting function $W_2(s)$ in H_∞ synthesis to find a stabilizing controller for the parametric uncertain system of Fig. 2. The constant MPR weighting shows much promise based on its simplicity and effective use of the well-developed H_∞ synthesis theory. The disadvantage of the MPR weighting approach is that unnecessary conservatism may be introduced in final controller design. This observation leads us to a consideration of weighting functions based on the critical direction theory.

3.2 The Critical Direction Theory

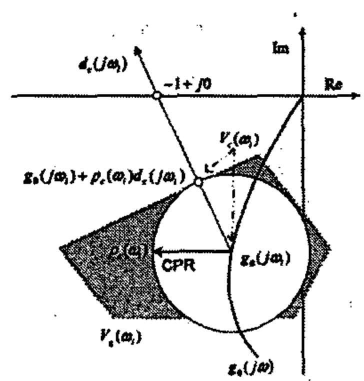

Figure 3: Symbols used in the critical direction theory.

For a given interval system $g(s)$ with assumptions *(B1)* and *(B2)*, the closed-loop system of Fig. 1 is stable if and only if the critical point $-1+j0$ is outside the value-set of $g(j\omega)$ at every frequency ω. Motivated by this critical point exclusion principle, Latchman *et al.* [1] [8] proposed a necessary and sufficient frequency-domain robustness analysis technique, namely, the critical direction theory, by recognizing that for the entire value-set at any given frequency, there is only one direction of perturbation of relevance to the stability analysis.

The solid curve in Fig. 3 describes the nominal system $g_0(j\omega)$ within a pre-specified frequency range. For a specific frequency point ω_i, the uncertainty family $g(j\omega_i)$ is plotted and characterized through the value-set $V_g(\omega_i) := \{g(j\omega_i) \mid g(j\omega_i) = g_0(j\omega_i) + \delta_g(j\omega_i, \epsilon), \epsilon \in \Delta(\epsilon), \delta_g(j\omega_i, \epsilon) = c(j\omega_i)\delta(j\omega_i, \epsilon)$. The critical direction is defined as

$$d_c(j\omega_i) := -\frac{1 + g_0(j\omega_i)}{|1 + g_0(j\omega_i)|}. \tag{4}$$

Accordingly, the family of uncertainty plants along the critical direction is defined as the critical value-set $V_c(\omega_i) := \{g_c(j\omega_i) \mid g_c(j\omega_i) = g_0(j\omega_i) + \alpha d_c(j\omega_i), \text{for some } \alpha \in R_+\}$. When the critical value-set $V_c(\omega_i)$ is convex, the critical perturbation radius (CPR) is defined as

$$\rho_c(\omega_i) := \max_{\alpha \in R_+} \{\alpha \mid g_c(j\omega_i) = g_0(j\omega_i) + \alpha d_c(j\omega_i) \in V_c(\omega_i)\}. \tag{5}$$

Finally, the Nyquist robust stability margin is defined as

$$k_N := \left\| \frac{\rho_c(\omega)}{|1 + g_0(j\omega)|} \right\|_\infty. \tag{6}$$

The necessary and sufficient stability condition is given in next theorem.

Theorem 1 *[8] Consider the uncertain system $g(s)$ given in Fig. 1 satisfying the assumptions (B1) and (B2) listed in Section 2. Then, under the assumption that the critical value-set is convex, the closed-loop system of Fig. 1 is stable if and only if $k_N < 1$.*

Note that as in [8], the previous description assumes that the critical value-set, not necessarily the whole value-set, is convex. More general work can be found in [1] where the critical direction theory is applied for the case of non-convex critical value-sets for more general systems with parametric uncertainties, and a new definition of the CPR is introduced to avoid conservatism. In this paper we use the definition given in (5) since as pointed out in [1] the critical perturbation radius computed by (5) is no worse than the MPR.

4 Main Results

4.1 *Sub-Robust* Controllers Synthesis

We first consider using a *dynamic* weighting method to reduce conservatism in determining the maximum stabilizable uncertainties, and then proceed to an iterative design for exactly calculating ϵ_{max}.

4.1.1 The Dynamic MPR Weighting:

In H_∞ synthesis, weighting functions are assumed to be stable and real rational transfer-functions; but, $\rho_m(\omega)$ is just a set of non-negative values over a specified frequency range. This situation is reminiscence of the well-known $D-K$ iteration in μ synthesis in which the frequency-dependent scalings $D(\omega)$ are approximated by a transfer function $\bar{D}(s)$ [9]. Similarly, by using a numerical inversion technique, a stable, minimum-phase, and real rational transfer-function approximation $\rho_m(s)$ to $\rho_m(\omega)$ can be found in the sense that

$$|\rho_m(j\omega)| = \rho_m(\omega), \ \forall\omega. \tag{7}$$

We call this approach *dynamic MPR weighting.* It is straightforward to verify that MPR is non-decreasing when the uncertainty level ϵ increases. Let $\epsilon_{sub-max}$ be the largest value of ϵ such that the system $p(s)$ in Fig. 1 is stabilizable by using only the MPR weighting approach. Then $\epsilon_{sub-max}$ can be found by increasing ϵ and testing the stability of H_∞ controllers synthesized using the corresponding dynamic MPR weighting functions. A computationally efficient algorithm can be designed using a bisection method.

Algorithm for Sub-Robust Controller Synthesis

- Step 1. Set the initial bounds of ϵ as $\epsilon_{lb} = 0, \epsilon_{ub} = \epsilon_u$, where ϵ_u is obtained based on the discussion in Section 2; and assign a value to the tolerance for the stopping criterion τ (typically between 10^{-2} and 10^{-4}).
- Step 2. Let $\epsilon = (\epsilon_{lb} + \epsilon_{ub})/2$. Check whether an admissible controller can be designed by merely using the MPR weighting approach.
  - If the interval system is stabilizable, $\epsilon_{lb} = \epsilon$, and iterate Step 2.
  - Else, $\epsilon_{up} = \epsilon$. Now if $(\epsilon_{ub} - \epsilon_{lb})/\epsilon_{lb} < \tau$, set $\epsilon_{sub-max} = (\epsilon_{lb} + \epsilon_{ub})/2$ and end the iteration. Else, *iterate Step 2.*

4.2 The ECPR and its Properties

To find the ϵ_{max}, namely to design a robust controller for which the closed-loop of Fig. 1 is marginally stable at ϵ_{max}, it is desirable to use an *exact* weighting function based on the necessary and sufficient stability condition associated with the critical direction theory. Assume that a robust controller for the system of Fig. 1 is $c_{rc}(s)$; then the effective critical perturbation radius (ECPR) is defined as

$$\rho_e(\omega) := \frac{\rho_c(\omega)}{|c_{rc}(j\omega)|} \tag{8}$$

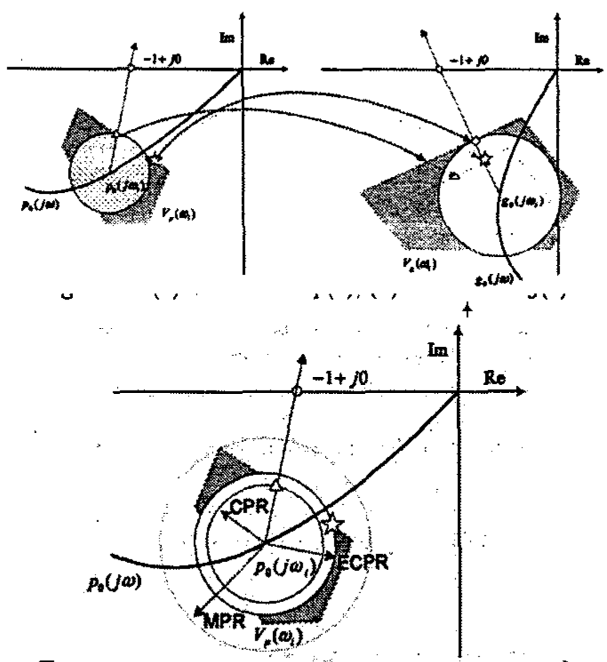

Figure 5: Illustration of the definition of the ECPR.

where $\rho_c(\omega)$ is the CPR of the cascaded system $g(j\omega) = c_{rc}(j\omega)p(j\omega)$ at frequency ω. Note that $\rho_c(\omega)$ is associated with the value-set of $g(j\omega) = c_{rc}(j\omega)p(j\omega)$, whereas $\rho_e(\omega)$ is associated with the value-set of $p(j\omega)$, as illustrated in Figures 4 and 5. The value of $\rho_e(\omega)$ at each frequency ω is just a non-negative number. The function $\rho_e(s)$ is used as an approximation function to $\rho_e(\omega)$, in the same fashion as $\rho_m(s)$ approximates $\rho_m(\omega)$.

Notice that $g(s) = c_{rc}(s)p(s)$. When multiplied by the controller $c_{rc}(s)$, the value-set of $p(j\omega)$ may be shifted, rotated, and/or magnified or contracted before producing the value-set of $g(j\omega)$. Fig. 4 illustrates the behavior of the corresponding variation caused by the controller $c_{rc}(s)$ between the value-set of $p(j\omega_i)$ and that of $g(j\omega_i)$ at frequency ω_i. Fig. 4 uses the following symbols: $p_\Delta(j\omega_i)$ is the intersection of the critical line originating from $p_0(j\omega_i)$ and the boundary of the value-set of $p(j\omega_i)$ as shown in Fig. 4 (a); $g_\diamond(j\omega_i)$ represents the intersection of the critical line originating from $g_0(j\omega_i)$ and the boundary of the value-set of $g(j\omega_i)$ as shown in Fig. 4 (b); and, $p_\star(j\omega_i)$ denotes the intersection of the boundary of the value-set of $p(j\omega_i)$ and a certain line segment originating from $p_0(j\omega_i)$ as shown in Fig. 4 (a). This line segment is shifted and rotated to coincide with the critical line in the value-set of $g(j\omega_i)$. It may be somewhat confusing to draw a shifted and rotated copy of the value-set of $p(j\omega_i)$ around the value-set of $g(j\omega_i)$; however, it is important to understand that it is the boundary point $p_\star(j\omega_i)$, not $p_\Delta(j\omega_i)$, that is critical for the stability analysis when a controller $c_{rc}(s)$ is cascaded with $p(s)$. The properties of the ECPR weight are summarized in the following theorems.

Theorem 2 *The function $\rho_e(s)$ is the* exact *weighting function in mixed-sensitivity H_∞ synthesis for designing a robust controller $c_{rc}(s)$.*

Proof: Following the CPR definition in (5),

$$\begin{aligned} \rho_c(j\omega) &= |c_{rc}(j\omega)p_\star(j\omega) - c_{rc}(j\omega)p_0(j\omega)| \\ &= |c_{rc}(j\omega)||p_\star(j\omega) - p_0(j\omega)| \\ &= |c_{rc}(j\omega)|\rho_e(\omega) \\ &= |c_{rc}(j\omega)||\rho_e(j\omega)| \end{aligned} \tag{9}$$

where $c_{rc}(j\omega)p_\star(j\omega)$ is the intersection of the critical line originating from $g_0(j\omega)$ and the boundary of the value-set of $g(j\omega)$ at frequency ω. According to Theorem 1, the system with controller $c_{rc}(s)$ is stable if and only if

$$k_N = \left\| \frac{\rho_c(\omega)}{|1 + g_0(j\omega)|} \right\|_\infty < 1. \tag{10}$$

Since the robust controller $c_{rc}(s)$ is defined as the marginally admissible controller at the maximum stability margin ϵ_{max}, the following equality exists,

$$k_N = \left\| \frac{|c_{rc}(j\omega)||\rho_e(j\omega)|}{|1 + c_{rc}(j\omega)p_0(j\omega)|} \right\|_\infty = 1 \tag{11}$$

where $\rho_e(s)$ is associated with the value-set of $p(s)$. Therefore, $\rho_e(s)$ is the *exact* weighting function in the sense of designing a robust controller $c_{rc}(s)$. ∎

Theorem 3 *The exact weighting function $\rho_e(s)$ satisfies*

$$\|\rho_e(j\omega)\|_\infty = \max_\omega \{\rho_e(\omega)\} \geq \frac{1}{\|T_{ur}(j\omega)\|_\infty}, \tag{12}$$

where $T_{ur}(s) = \frac{c_{rc}(s)}{1+c_{rc}(s)p_0(s)}$.

Proof: Since the robust controller $c_{rc}(s)$ achieves marginal stability, using the standard H_∞ notation, the following inequality holds,

$$\rho(X_\infty Y_\infty) \leq \gamma^2, \tag{13}$$

where $\gamma = 1$ and

$$\|T_{zr}(s)\|_\infty \leq \gamma = 1. \tag{14}$$

Invoking the definition of $T_{zr}(s)$ and replacing $W_2(s)$ with the ECPR yields

$$\begin{aligned} \|T_{zr}(j\omega)\|_\infty &= \|T_{ur}(j\omega)W_2(j\omega)\|_\infty \\ &= \|T_{ur}(j\omega)\rho_e(j\omega)\|_\infty = 1. \end{aligned} \tag{15}$$

According to the inequality property of a general norm,

$$\|T_{ur}(j\omega)\|_\infty \|\rho_e(j\omega)\|_\infty \geq \|T_{ur}(j\omega)\rho_e(j\omega)\|_\infty = 1, \quad (16)$$

therefore

$$\|\rho_e(j\omega)\|_\infty = \max_\omega \{\rho_e(\omega)\} \geq \frac{1}{\|T_{ur}(j\omega)\|_\infty}. \quad (17)$$

■

Theorem 4 *The function $\rho_e(s)$ and the associated non-conservative controller $c_{rc}(s)$ satisfy $k_N = 1$ and $\|T_{zr}(s)\|_\infty = 1$.*

Proof: These properties can be easily extracted from theorems 2 and 3. ■

4.3 Robust Controllers Synthesis: Controller Tuning

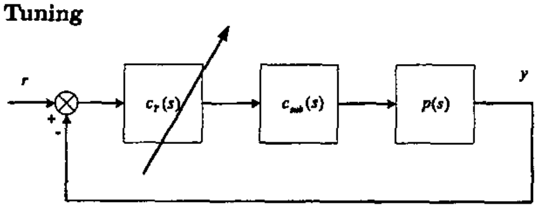

Figure 6: The tuning process using a controller $c_t(s)$.

Referring to the equality (9), it is clear that conservatism may be introduced using the over-bounding weight $\rho_m(s)$ instead of $\rho_e(s) = |p_*(j\omega) - p_0(j\omega)|$. Therefore, even though the interval system is unstabilizable by the MPR weighting approach at $\epsilon > \epsilon_{sub-max}$, there may still exist other weighting functions which will result in an admissible controller. A non-conservative robust controller should be designed using the necessary and sufficient weighting function $\rho_e(s)$. Unfortunately, the ECPR is defined only after the robust controller has been specified. To resolve this seemingly contradictory situation, we propose a tuning method to find the maximum stability margin ϵ_{max} and an associated robust controller $c_{rc}(s)$.

Given a controller $c(s)$, the following notation is used:

$$\tilde{\rho}_e(\omega) = \frac{\rho_c(\omega)}{|c(j\omega)|}. \quad (18)$$

Note that $\tilde{\rho}_e(\omega)$ is equivalent to the ECPR defined in (8), but here $c(s)$ is not restricted to the robust controller $c_{rc}(s)$. Hence, different ECPR weights exist for different controllers $c(s)$. Now recall that the sub-maximal stabilizable uncertainty $\epsilon_{sub-max}$ was found from the sub-robust controller synthesis algorithm. Suppose that $c_{sub}(s)$ is one of the non-admissible controllers synthesized by the MPR approach at a certain uncertainty level ϵ_k that is greater than $\epsilon_{sub-max}$. Then a simple but effective tuning approach can be implemented by cascading $c_{sub}(s)$ with the following simple lead-lag controller $c_t(s)$ with tuning parameters k and z:

$$c_t(s) = k\,\frac{s+z}{s+p}. \quad (19)$$

To reduce the computation complexity, the parameter p is taken as a constant chosen arbitrarily in the range $1 \leq p \leq 20$. In the sequel we use $p = 10$.

Inserting $c_t(s)$ into the loop, a new tuned-ECPR is calculated as

$$\tilde{\rho}_{e,t}(\omega) = \frac{\rho_c(\omega)}{|c_t(j\omega)c_{sub}(j\omega)|} = |p_t(j\omega) - p_0(j\omega)| \quad (20)$$

where $p_t(j\omega)$ is a certain boundary point in the neighborhood of $p_*(s)$ (see Fig. 4). Note that now $g(s) = c_t(s)c_{sub}(s)p(s)$.

Hence, generally, a new weighting function $\tilde{\rho}_{e,t}(s)$ is obtained by tuning $c_t(s)$. For each $\tilde{\rho}_{e,t}(s)$ resulting from varying the tuning parameters k and z, a new controller $c(s)$ can be designed by using the H_∞ synthesis tool. For each c(s), the Nyquist robust stability margin k_N of the system $g(s)$ (*i.e.*, $p(s)c(s)$) is calculated and checked. Whenever $k_N < 1$, the uncertainty level ϵ_k is stabilizable by using the ECPR weight. The desired parameters k and z are those for which the corresponding synthesized controller $c(s)$ results in the minimum k_N.

Algorithm for Robust Controller Synthesis

Step 1: Let $c_{sub}(s)$ be the controller synthesized by using the MPR approach at a certain value of $\epsilon_k > \epsilon_{sub-max}$. Define $c_t(s) = k\,\frac{s+z}{s+10}$.

Step 2: Fix z at 10, and vary k within a suitable range about unity, then calculate $\tilde{\rho}_{e,t}(s)$ of the system $g(s) = c_t(s)c_{sub}(s)p(s)$. Synthesize a controller to replace $c_{sub}(s)$ using H_∞ synthesis with $\tilde{\rho}_{e,t}(s)$. Assign $c_{sub}(s)$ to $c(s)$ in Fig. 1, and calculate k_N. Let k_{rc} equal the value of k which corresponds to the minimal value of k_N.

Step 3: Fix k at k_{rc}, and vary factor z within the vicinity of 10, then calculate $\tilde{\rho}_{e,t}(s)$ of the system $g(s) = c_t(s)c_{sub}(s)p(s)$. Synthesize a controller to replace $c_{sub}(s)$ using H_∞ synthesis with $\tilde{\rho}_{e,t}(s)$. Assign $c_{sub}(s)$ to $c(s)$ in Fig. 1, and calculate k_N. Let z_{rc} equal the value of z which corresponds to the minimal value of k_N. If the minimal value of k_N is less than unity, a robust controller $c_{rc}(s)$ is found with the tuning parameters k_{rc} and z_{rc}; else the algorithm should be restarted with $\epsilon_{sub-max} < \epsilon < \epsilon_k$.

The above tuning procedure involves a step-by-step process, searching alternatively over parameters k

and z. A global search over the parameters k and z is also possible, but is much more demanding. Our preliminary experience suggests that the simple heuristic algorithm proposed above works quite well as illustrated in the following example.

5 An Example

Consider the interval plant

$$p(s, \mathbf{q}) = \frac{5s + q_1}{s^2 + q_2 s + q_3};$$

where,

$$q_1 \in [4-\epsilon, 4+\epsilon], q_2 \in [2-\epsilon, 2+\epsilon], q_3 \in [-15-\epsilon, -15+\epsilon].$$

Note that to satisfy Assumption *(A)* in Section 2, $\epsilon_u = 7.5$ [2] [4].

Following the algorithm for sub-robust controller design given in Section 4.1, the sub-maximum stability margin is found to be $\epsilon_{sub-max} = 6.2499$.

At $\epsilon > 6.2499$, the interval system appears unstabilizable when just using the MPR weighting. However, it is possible for us to find a larger margin by starting the tuning process with the controller synthesized by the MPR weighting at $\epsilon = 7$, namely,

$$c_{sub}(s) = \frac{n_{sub}(s)}{d_{sub}(s)};$$

$$n_{sub}(s) = 16981.5925s^4 + 229086.4782s^3 + 1106292.6284s^2 + 2118935.9323s + 959678.4155$$

$$d_{sub}(s) = s^5 + 3377.3566s^4 - 732.4843s^3 - 162599.2866s^2 - 565429.0321s - 296531.9683.$$

Following the tuning process of the algorithm for robust controller synthesis, the values $k = 0.4$, and $z = 11.6$ are found to result in a robust controller called $c_{rc}(s)$ with which the system in Fig. 1 with $\epsilon = 7$ has $k_N = 0.9593$. And,

$$c_{rc}(s) = \frac{n_{rc}(s)}{d_{rc}(s)}; \qquad (21)$$

$$n_{rc}(s) = 134877972s^4 + 127457.5715s^3 + 372751.7843s^2 + 374990.3134s + 58480.1182$$

$$d_{rc}(s) = s^5 + 581.2072s^4 - 29230.1744s^3 - 131929.4094s^2 - 143703.8491s - 24276.0372.$$

The stability margin of the system in Fig. 1 with controller $c_{rc}(s)$ (21) is found to be $\epsilon_{max} = 7.1796$, which is very close to the upper bound of the uncertainty level of 7.5. Note that assumptions *(B1)* and *(B2)* are satisfied for $c_{rc}(s)$ (21) at ϵ up to ϵ_u.

6 Conclusion

The main contribution of this paper is to introduce necessary and sufficient weighting functions and to use the H_∞ method in designing a non-conservative robust controller for systems with parametric uncertainties. A sub-robust controller synthesis algorithm enhanced by a bisection algorithm is constructed and then a tuning procedure is established to find the maximum stabilizable uncertainty. The simulation results demonstrate that this paper proposes a very promising strategy in synthesizing robust controllers for systems with parametric uncertainties. The future work involves the refinement of the tuning procedure with a view to formulating a convex optimization problem.

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