A Restatement of the Major Principal Direction Alignment Principle for Robustness Quantification

C. T. Baab[1], H. A. Latchman[2], and O. D. Crisalle[1*]

[1] *Department of Chemical Engineering*
University of Florida
Gainesville, FL 32611-6005

[2] *Department of Electrical Engineering*
University of Florida
Gainesville, FL 32611

crisalle@che.ufl.edu

Abstract

The major principal direction alignment (MPDA) principle, developed in the context of robust control theory, states that a necessary and sufficient condition for the spectral radius of a matrix to equal its maximum singular value is that the major input and the major output principal directions of the matrix be aligned. The MPDA principle emerged from a study of the derivatives of the maximum singular value. An ambiguity that occurs when the maximum singular value is repeated is considered in this paper, together with a modified statement of the major principal direction alignment principle. The new necessary and sufficient condition for the spectral radius of a matrix to equal its maximum singular value is that there exists at least one major input and output principal direction pair of the matrix that is aligned. A rigorous proof is provided for the new necessary and sufficient condition, which makes use of early results on dual norms and dual vectors. An example is presented to illustrate the results.

1. Introduction

The structured singular value for scalar, complex and diagonal uncertainty blocks is defined as

$$\mu(\mathbf{M}) = \sup_{\mathbf{U} \in \mathcal{U}} \rho(\mathbf{MU}) \qquad (1)$$

where $\mathbf{M} \in C^{n \times n}$,

$$\mathcal{U} := \left\{ diag(e^{j\theta_1}, e^{j\theta_2}, \cdots, e^{j\theta_n}) \middle| 0 \le \theta_i < 2\pi, i = 1,2,\cdots,n \right\},$$

and ρ is the spectral radius [1], is a widely accepted tool in the analysis of robustness properties of linear systems. In general, an arbitrary interconnection of systems and uncertainties can be rearranged into the block-diagonal standard form after an appropriate augmentation of the uncertainty model to reflect the underlying uncertainty structure. Calculating μ is not trivial; in fact the optimization implied by (1) has been proven to be NP-hard [2]. The difficulty is that the spectral radius is non-convex over the set of (block) unitary matrix transformations. One approach is to consider upper bounds for the spectral radius that can be calculated easily, and ideally these bounds should be attainable to eliminate conservatism. The maximum singular value is a reasonable choice for an upper bound because it is invariant under unitary matrix transformations. In addition the maximum singular value upper bound can be decreased by optimizing over appropriately defined similarity transformations, because the spectral radius is invariant under such transformations. Ultimately, the problem becomes one of conditioning a matrix through optimal similarity and unitary transformations to achieve equality between the spectral radius and maximum singular value. Consequently, determining the conditions under which the upper bound is attained is a significant issue in the field of robust control. Much attention has been given in the last two decades to the structural relationship between the upper bound and the spectral radius. The doctoral dissertation theses [10] and [11] are excellent exponent of the pivotal results in the field, and are part of a vast body of literature.

In addressing the existence of solutions to the problem (1), Kouvaritakis and Latchman introduced the *major principal direction alignment* (MPDA) property [3]. The result states that the spectral radius of a matrix is equal to the maximum singular value of the matrix if and only if the major input and the major output principal directions of the matrix are aligned. MPDA is a strict condition for a matrix, but can be used to determine the optimal positive diagonal matrix and unitary matrix that results in equality between the aforementioned definition of μ and the maximum singular value upper bound. The proof of the MPDA principle is based on linear algebra arguments, and considers separately the cases of a unique and a repeated maximum singular value. The proof of necessity and

* Author to whom all correspondence should be addressed.

sufficiency as presented in [3] is quite straightforward for the case of a unique maximum singular value, but leaves room for a possibly ambiguity of interpretation for the case of a repeated maximum singular value. This paper explores the statement of the MPDA principle for a repeated maximum singular value and presents a revised formulation of the result to remove all potential for ambiguity. In addition, a new proof of the MPDA result is presented that is based on the properties of dual norms and dual vectors which actually served as the basis for the original MPDA result.

The mathematical concepts of dual norms and dual vectors are utilized by Bauer [4] and by Stoer and Witzgall [5] in their early work on determining when the spectral radius equals the maximum singular value for positive matrices transformed by non-negative diagonal matrices. One motivation for the focus on positive matrices is that they have good numerical properties (*i.e.*, introduce less round-off error), and therefore may be successfully used for conditioning of matrices. In addition, positive matrices remain positive under transformations by non-negative diagonal matrices, leading to connections with the Perron-roots $\pi(\mathbf{M})$ (positive eigenvalues of largest modulus) of positive matrices $\mathbf{M}$ [6]. One contribution of this paper is to show that the MPDA principle may be interpreted as an elegant extension of the dual vector/dual norm results of the classical work of Bauer and of Stoer and Witzgall on non-negative real matrices, to the case of general complex matrices.

To facilitate the exposition that follows, the next section provides relevant mathematical background, including a discussion of the singular value decomposition, a summary of dual-norm and dual-vector concepts, and a dual eigenvector result. Section 3 revisits the original MPDA theorem. Section 4 provides a modified statement of the MPDA principle theorem that explicitly considers a repeated maximum singular value, along with a proof based on the dual-norm and dual-vector theory. Several examples are given in Section 5 and concluding remarks are made in Section 6.

2. Mathematical Background

2.1. The Singular Value Decomposition and Eigenvalue Decomposition

Without loss of generality, in this paper only square matrices are considered. This follows from the well-known fact that the structured singular value is not affected when a rectangular matrix is augmented with rows or columns of zeros to provide a square matrix. The singular values of square matrix $\mathbf{A} \in C^{n \times n}$ are given by

$$\sigma_i(\mathbf{A}) := \sqrt{\lambda_i(\mathbf{A}^*\mathbf{A})}, \quad i = 1,2,\cdots,n$$

where $\lambda_i(\mathbf{A}^*\mathbf{A})$ represent the i-th eigenvalues of the matrix $\mathbf{A}^*\mathbf{A}$ and where the singular values are ordered such that

$$\sigma_1(\mathbf{A}) \geq \sigma_2(\mathbf{A}) \geq \cdots \geq \sigma_n(\mathbf{A})$$

The singular value decomposition of an arbitrary matrix $\mathbf{A} \in C^{n \times n}$ is given by

$$\mathbf{A} = \mathbf{X}(\mathbf{A})\Sigma(\mathbf{A})\mathbf{Y}^*(\mathbf{A}) \tag{2}$$

where $\Sigma(\mathbf{A}) := diag(\sigma_1(\mathbf{A}), \sigma_2(\mathbf{A}), \cdots, \sigma_n(\mathbf{A}))$ is the diagonal matrix of singular values organized in descending order, and $\mathbf{X}(\mathbf{A})$ and $\mathbf{Y}(\mathbf{A})$ are unitary matrices. The matrices $\mathbf{X}(\mathbf{A})$ and $\mathbf{Y}(\mathbf{A})$ are of the form

$$\mathbf{X}(\mathbf{A}) = [\mathbf{x}_1(\mathbf{A}) \quad \mathbf{x}_2(\mathbf{A}) \quad \cdots \quad \mathbf{x}_n(\mathbf{A})]$$

$$\mathbf{Y}(\mathbf{A}) = [\mathbf{y}_1(\mathbf{A}) \quad \mathbf{y}_2(\mathbf{A}) \quad \cdots \quad \mathbf{y}_n(\mathbf{A})]$$

where the set of normalized left singular-vectors (*input principal directions*) $\{\mathbf{x}_i(\mathbf{A})\}$ and normalized right singular-vectors (*output principal directions*) $\{\mathbf{y}_i(\mathbf{A})\}$ for $i = 1,2,\cdots,n$, respectively constitute orthonormal eigenbasis of $\mathbf{AA}^*$ and $\mathbf{A}^*\mathbf{A}$. Furthermore, a pair of singular vectors $\{\mathbf{x}_i(\mathbf{A}), \mathbf{y}_i(\mathbf{A})\}$ is associated with each singular value $\sigma_i(\mathbf{A})$ through the relationship

$$\mathbf{Ay}_i(\mathbf{A}) = \sigma_i(\mathbf{A})\mathbf{x}_i(\mathbf{A}) \tag{3}$$

The maximum singular value is denoted $\overline{\sigma}(\mathbf{A})$. It must be noted that the maximum singular value can be repeated, *i.e.* $\overline{\sigma}(\mathbf{A}) = \sigma_1(\mathbf{A}) = \sigma_2(\mathbf{A}) = \cdots$. A maximum left/right singular vector pair (or major output/input principal direction pair) $\{\overline{\mathbf{x}}(\mathbf{A}), \overline{\mathbf{y}}(\mathbf{A})\}$ is any pair of left and right singular vectors $\mathbf{x}_i(\mathbf{A})$ and $\mathbf{y}_i(\mathbf{A})$ that corresponded to the maximum singular value and satisfies (3). Necessarily, a major output principal direction and a major input principal direction respectively must be elements of the orthonormal eigensubspace of $\mathbf{AA}^*$ and $\mathbf{A}^*\mathbf{A}$ associated with the maximum singular value.

In this paper the following definitions are used in relation to the eigenvalue decomposition [7], [8], [9]. Let $\lambda_i(\mathbf{A})$ be and eigenvalue of $\mathbf{A}$, then a right eigenvector $\mathbf{v}_i$ of $\mathbf{A}$ is any non-zero vector that satisfies

$$\mathbf{Av}_i = \lambda\mathbf{v}_i$$

Furthermore, a left eigenvector $\mathbf{w}_i$ of $\mathbf{A}$ is any non-zero vector that satisfies

$$\mathbf{w}_i^*\mathbf{A} = \lambda\mathbf{w}_i^*$$

The reader is cautioned that some authors use the term left eigenvector to denote an eigenvector of $\mathbf{A}^T$. Finally, an eigenvalue of maximum modulus is any eigenvalue $\lambda_i(\mathbf{A})$ that satisfies $|\lambda_i(\mathbf{A})| = \rho(\mathbf{A})$.

2.2. Dual Norms and Dual Vectors

The mathematical concepts of dual norms and dual vectors utilized in this section are explained in [4] and are succinctly reviewed here to facilitate the theoretical developments. Given a vector norm $\|\cdot\|$, its dual vector norm $\|\cdot\|_D$ is defined as

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$$\|\mathbf{y}\|_D := \max_{\|\mathbf{x}\|=1} \mathrm{Re}\,\mathbf{y}^*\mathbf{x} = \max_{\|\mathbf{x}\|\neq 0} \frac{\mathrm{Re}\,\mathbf{y}^*\mathbf{x}}{\|\mathbf{x}\|}$$

For such dual norms the Holder inequality

$$\|\mathbf{y}\|_D \|\mathbf{x}\| \geq \mathrm{Re}\,\mathbf{y}^*\mathbf{x}$$

holds and is sharp, *i.e.*, for any $\mathbf{y}_0$ there exists at least one $\mathbf{x}_0$, and for any $\mathbf{x}_0$ there exists at least one $\mathbf{y}_0$ such that the equality holds [4]. If such a pair $(\mathbf{x}_0, \mathbf{y}_0)$ with $\|\mathbf{y}_0\|_D \|\mathbf{x}_0\| = \mathrm{Re}\,\mathbf{y}_0^*\mathbf{x}_0$ also satisfies the scaling condition

$$\|\mathbf{y}_0\|_D \|\mathbf{x}_0\| = 1$$

it is called a *dual pair*. Note that the dual vector of $\mathbf{x}$ is often written $(\mathbf{x})_D$. A pair $(\mathbf{x}_0, \mathbf{y}_0)$ is *strictly dual* and is written $\mathbf{y}_0 \|_D \mathbf{x}_0$ if $\|\mathbf{y}_0\|_D \|\mathbf{x}_0\| = \mathbf{y}_0^*\mathbf{x}_0 = 1$. For strictly homogenous norms (*i.e.*, those satisfying $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$ for all complex scalars α) the Holder inequality may be sharpened to [4]

$$\|\mathbf{y}\|_D \|\mathbf{x}\| \geq |\mathbf{y}^*\mathbf{x}|$$

For a dual pair $(\mathbf{x}_0, \mathbf{y}_0)$ under a homogenous norm it follows that $\mathrm{Re}\,\mathbf{y}_0^*\mathbf{x}_0 = \|\mathbf{y}_0\|_D \|\mathbf{x}_0\| \geq |\mathbf{y}_0^*\mathbf{x}_0|$, which implies that $\mathrm{Re}\,\mathbf{y}_0^*\mathbf{x}_0 = \mathbf{y}_0^*\mathbf{x}_0$. Hence, for a strictly homogenous norm every pair of dual vectors $(\mathbf{x}_0, \mathbf{y}_0)$ is also a strictly dual pair. In addition, there exists a strict dual $\mathbf{y}_0$ for any $\mathbf{x}_0 \neq 0$, and a strict dual $\mathbf{x}_0$ for any $\mathbf{y}_0 \neq 0$. Furthermore, the concept of *approximately dual* vectors is proposed such that a pair $(\mathbf{x}_0, \mathbf{y}_0)$ is said to be *approximately dual* if $\|\mathbf{y}_0\|_D \|\mathbf{x}_0\| = |\mathbf{y}_0^*\mathbf{x}_0| = 1$.

In general, the dual norm of a p-norm $\|\mathbf{x}\|_p := (\sum |x_i|^p)^{1/p}$ is the associated q-norm $\|\cdot\|_q$, where $1/p + 1/q = 1$. So, the infinity-norm and the 1-norm are duals, and the dual norm of the (Euclidean) 2-norm is itself. For the 2-norm, a pair $(\mathbf{x}_0, \mathbf{y}_0)$ is dual if $\mathbf{y}_0 = \mathbf{x}_0 / \|\mathbf{x}_0\|_2^2$, and is approximately dual if $\mathbf{y}_0 = e^{j\theta}\mathbf{x}_0 / \|\mathbf{x}_0\|_2^2$.

2.3. Dual Eigenvector Result

The basis of the following lemma is a result of Bauer [4] on the field of values of a matrix.

Lemma 1. *If the spectral radius of a matrix* $\mathbf{A} \in C^{n\times n}$ *is equal to the maximum singular value of* $\mathbf{A}$, *then for each normalized right eigenvector* $\mathbf{v}_i$ *associated with an eigenvalue of maximum modulus* $\lambda_i(\mathbf{A})$ *there exists a normalized left eigenvector* $\mathbf{w}_i = \mathbf{v}_i$ *such that* $\mathbf{v}_i$ *and* $\mathbf{w}_i$ *form a dual pair* $\mathbf{w}_i \|_D \mathbf{v}_i$.

Proof. Lemma 1 is a specialization of Bauer's results [4] to the case of the Euclidean norm which permits expressing the claim in terms of the maximum singular value of the matrix. Bauer's proof makes use of dual-norm dual-vector theory presented earlier.

2.4. Eigenvector-Singular Vector Equivalence Result

The following result is a consequence of Lemma 1.

Lemma 2. *If the spectral radius of a matrix* $\mathbf{A} \in C^{n\times n}$ *is equal to the maximum singular value of* $\mathbf{A}$, *then each normalized right eigenvector* $\mathbf{v}_i$ *of* $\mathbf{A}$ *associated with an eigenvalue of maximum modulus* $\lambda_i(\mathbf{A})$ *is also a right singular vector* $\mathbf{y}_i$ *of* $\mathbf{A}$ *associated with the maximum singular value* $\bar{\sigma}(\mathbf{A})$.

Proof. If suffices to prove that $\mathbf{v}_i$ is a right eigenvector of $\mathbf{A}^*\mathbf{A}$ associated with an eigenvalue whose square root is $\bar{\sigma}(\mathbf{A})$. The sufficiently claim holds because by definition the right singular vectors $\mathbf{y}_i$ are an orthonormal eigenbasis of $\mathbf{A}^*\mathbf{A}$ and the singular values are the square roots of the eigenvalues of $\mathbf{A}^*\mathbf{A}$. First, from Lemma 1 it follows that for each normalized right eigenvector $\mathbf{v}_i$ of $\mathbf{A}$ associated with an eigenvalue of maximum modulus $\lambda_i(\mathbf{A})$ there exists a normalized left eigenvector $\mathbf{w}_i = \mathbf{v}_i$ of $\mathbf{A}$. For any such eigenvector $\mathbf{v}_i$

$$\begin{aligned}\mathbf{A}^*\mathbf{A}\mathbf{v}_i &= \mathbf{A}^*\mathbf{v}_i\lambda_i(\mathbf{A}) = (\mathbf{v}_i^*\mathbf{A})^*\lambda_i(\mathbf{A}) \\ &= (\mathbf{w}_i^*\mathbf{A})^*\lambda_i(\mathbf{A}) = (\lambda_i(\mathbf{A})\mathbf{w}_i^*)^*\lambda_i(\mathbf{A}) \\ &= \mathbf{v}_i^{**}\lambda_i^*(\mathbf{A})\lambda_i(\mathbf{A}) = |\lambda_i(\mathbf{A})|^2\mathbf{v}_i = \lambda_i(\mathbf{A}^*\mathbf{A})\mathbf{v}_i\end{aligned} \tag{4}$$

Hence, from (4) if follows that $\mathbf{v}_i$ is and eigenvector of $\mathbf{A}^*\mathbf{A}$ with eigenvalue $\lambda_i(\mathbf{A}^*\mathbf{A})$. Finally, $\sqrt{\lambda_i(\mathbf{A}^*\mathbf{A})} = \sqrt{|\lambda_i(\mathbf{A})|^2} = \sqrt{\bar{\sigma}^2(\mathbf{A})} = \bar{\sigma}(\mathbf{A})$, completing the proof.

3. Statement of the Major Principal Direction Alignment Property

In solving various robust control problems it is desirable to determine the conditions under which the spectral radius of a matrix attains its maximum singular-value upper bound. The major principal direction alignment (MPDA) property addresses this problem. Consider the singular value decomposition of a matrix $\mathbf{A}$ given by (2) where $\Sigma(\mathbf{A})$ is the diagonal matrix of singular values organized in descending order, and $\mathbf{X}(\mathbf{A})$ and $\mathbf{Y}(\mathbf{A})$ are unitary matrices whose columns are the respective output and input principal directions of $\mathbf{A}$, arranged in an order conformal with the order of the singular values. A *major* input principal direction $\bar{\mathbf{y}}(\mathbf{A})$ and a *major* output principal direction $\bar{\mathbf{x}}(\mathbf{A})$ of a matrix $\mathbf{A}$ are any input and output principal directions, respectively, corresponding to the maximum singular value $\bar{\sigma}(\mathbf{A})$ of $\mathbf{A}$. In addition, the major input principal-direction $\bar{\mathbf{y}}(\mathbf{A})$ and

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the major output principal-direction $\overline{\mathbf{x}}(\mathbf{A})$ are said to be *aligned* if the exists a real scalar $\theta \in R$ such that $\overline{\mathbf{y}}(\mathbf{A}) = e^{j\theta}\overline{\mathbf{x}}(\mathbf{A})$. The following statement of the Major Principal Direction Alignment (MPDA) property is reproduced from [3].

Theorem 1. *The spectral radius of any matrix* $\mathbf{A} \in C^{n\times n}$ *is equal to the maximum singular value of* $\mathbf{A}$ *if and only if the major input and output principal directions of* $\mathbf{A}$ *are aligned.*

Proof. The proof consists of two cases, namely, when the maximum singular value is distinct and when it is repeated. Details are found in [3].

For the case of a distinct maximum singular value, Theorem 1 is entirely accurate and unambiguous as stated, and the proof rigorous. Unfortunately, when there is a repeated maximum singular value the statement of the problem is less precise. The fact that the maximum singular value may have a multiplicity greater than one is not explicitly addressed in the statement of Theorem 1 and this leads to potential ambiguities in the interpretation of the result. For example, the statement of the theorem seems to suggest that every major input and output principal direction pair should be aligned. Instead, in the sequel, it is shown that it is only necessary that there be one such pair that is aligned. In other words, when the maximum singular value is repeated, there may exist some major input and output principal direction pairs that are not aligned when the spectral radius equals the maximum singular value. Examples to illustrate these phenomena are given in an ensuing section. A modified statement of MPDA with a proof based on duality arguments is provided in the next section.

4. Modified Statement of the Major Principal Direction Alignment Principle

The following theorem is a restatement of the MPDA principle in terms of dual vectors, reformulated to accurately take into account the case of a repeated maximum singular value.

Theorem 2. *The spectral radius of any matrix* $\mathbf{A} \in C^{n\times n}$ *is equal to the maximum singular value of* $\mathbf{A}$ *if and only if there exists a major input and major output principal direction pair of* $\mathbf{A}$ *that is approximately dual with respect to the Euclidean norm.*

Proof. First it is shown that the approximate duality of a major-input/major-output principal direction pair with respect to the Euclidean norm is equivalent to alignment of the pair. By definition, the pair is approximately dual with respect to the Euclidean norm if and only if

$$\overline{\mathbf{x}} = \frac{1}{\|\overline{\mathbf{y}}\|_2^2} e^{j\theta}\overline{\mathbf{y}} \qquad (5)$$

Principal directions are always normalized to have a unit norm, therefore (5) is equivalent to

$$\overline{\mathbf{x}} = e^{j\theta}\overline{\mathbf{y}}$$

which is exactly the condition for alignment.

Next, to prove sufficiency note that alignment of a major-input and major-output principal direction pair of $\mathbf{A}$ implies

$$\overline{\mathbf{y}}(\mathbf{A}) = e^{j\theta}\overline{\mathbf{x}}(\mathbf{A}) \qquad (6)$$

Pre-multiplication of equation (6) by $\mathbf{A}$ gives

$$\mathbf{A}\overline{\mathbf{y}}(\mathbf{A}) = e^{j\theta}\mathbf{A}\overline{\mathbf{x}}(\mathbf{A}) \qquad (7)$$

Using the singular value decomposition property

$$\mathbf{A}\overline{\mathbf{y}}(\mathbf{A}) = \overline{\sigma}(\mathbf{A})\overline{\mathbf{x}}(\mathbf{A}) \qquad (8)$$

and combining equation (7) and equation (8) gives

$$\mathbf{A}\overline{\mathbf{x}}(\mathbf{A}) = e^{-j\theta}\overline{\sigma}(\mathbf{A})\overline{\mathbf{x}}(\mathbf{A})$$

so that $\lambda = e^{-j\theta}\overline{\sigma}(\mathbf{A})$ emerges as an eigenvalue, of $\mathbf{A}$ with an associated eigenvector $\overline{\mathbf{x}}(\mathbf{A})$. Noting that the magnitude of the eigenvalues of $\mathbf{A}$ are always bounded from above by $\overline{\sigma}(\mathbf{A})$, if follows that

$$|\lambda| = \rho(\mathbf{A}) = \overline{\sigma}(\mathbf{A})$$

To prove necessity, assume $\rho(\mathbf{A}) = \overline{\sigma}(\mathbf{A})$ and invoke Lemma 2 to conclude that any right eigenvector $\mathbf{v}_i$ of $\mathbf{A}$ associated with an eigenvalue of maximum modulus $\lambda_i(\mathbf{A})$ is also a right singular vector $\mathbf{y}_i$ of $\mathbf{A}$ associated with the maximum singular value $\overline{\sigma}(\mathbf{A})$. Then, using (3), the corresponding left singular vectors are

$$\mathbf{x}_i(\mathbf{A}) = \frac{\mathbf{A}\mathbf{y}_i(\mathbf{A})}{\overline{\sigma}(\mathbf{A})} = \frac{\mathbf{A}\mathbf{v}_i(\mathbf{A})}{|\lambda_i(\mathbf{A})|} = \frac{\lambda_i(\mathbf{A})\mathbf{v}_i(\mathbf{A})}{|\lambda_i(\mathbf{A})|}$$
$$= e^{j\arg(\lambda_i(\mathbf{A}))}\mathbf{y}_i(\mathbf{A}) = e^{j\theta}\mathbf{y}_i(\mathbf{A})$$

Therefore, for each orthonormalized right-eigenvector $\mathbf{v}_i$ there is a major input/output principal direction pair that is aligned. Namely

$$\mathbf{y}_i(A) = \mathbf{v}_i$$

and

$$\mathbf{x}_i(\mathbf{A}) = e^{j\theta}\mathbf{y}_i(\mathbf{A}) \qquad (9)$$

are aligned, where

$$\theta = \arg(\lambda_i(\mathbf{A})) \qquad (10)$$

Finally, there is always at least one right eigenvector $\mathbf{v}_i$ of $\mathbf{A}$ associated with an eigenvalue of maximum modulus $\lambda_i(\mathbf{A})$; therefore, there must exist at least one major input/output principal direction pair that is aligned, which completes the proof.

Theorem 2 is a precise statement of the MPDA property. The theorem eliminates any ambiguity that may result when applying the MPDA property to the case of repeated maximum singular values. In addition, the proof makes use of earlier work on dual vectors and dual norms. One example is provided to illustrate the results of the paper.

5. Example

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -0.9026-1.0077i & 0.2586-0.1506i & 0.1661+0.2372i \\ 0.6086+0.2053i & 1.2588-1.1670i & -0.6442+0.2239i \\ 0.6487+0.2968i & -0.5918-0.4665i & 0.1641-1.4383i \end{bmatrix}$$

with eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -0.0687+0.1159i \\ -0.8719+0.2183i \\ 0.3920+0.1421i \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0.1807+0.1816i \\ 0.3478-0.2425i \\ 0.8670+0.0538i \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 0.6834-0.0523i \\ -0.3628-0.1105i \\ -0.4576-0.4208i \end{bmatrix}$$

and associated eigenvalues

$$\{\lambda_1, \lambda_2, \lambda_3\} = \left\{2.0000e^{-0.6000j}, 1.2503e^{-1.6660j}, 1.5996e^{-2.2555j}\right\}$$

and singular value decomposition $\mathbf{A} = \mathbf{X}\Sigma\mathbf{Y}^*$, with $\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3]$ and $\mathbf{Y} = [\mathbf{y}_1 \quad \mathbf{y}_2 \quad \mathbf{y}_3]$, where

$$\mathbf{x}_1 = \begin{bmatrix} -0.0018+0.0876i \\ 0.2741+0.8117i \\ 0.2903-0.4173i \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0.4121+0.3962i \\ -0.4542+0.0343i \\ -0.4806-0.4845i \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} 0.4426+0.6853i \\ 0.2421-0.0061i \\ 0.2491+0.4624i \end{bmatrix}$$

$$\mathbf{y}_1 = \begin{bmatrix} 0.1556 \\ -0.2965+0.8731i \\ 0.3367-0.1101i \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -0.7481 \\ -0.0272-0.1299i \\ 0.5404-0.3615i \end{bmatrix} \tag{11}$$

$$\mathbf{y}_3 = \begin{bmatrix} -0.6451 \\ -0.0400+0.3612i \\ -0.5455+0.3927i \end{bmatrix}$$

and with singular values

$$\{\sigma_1, \sigma_2, \sigma_3\} = \{2, 2, 1\}$$

Note that the spectral radius equals the maximum singular value, *i.e.*

$$|\lambda_1| = \rho(\mathbf{A}) = \bar{\sigma}(\mathbf{A}) = \sigma_1 = \sigma_2 = 2$$

In this case the eigenvalue of maximum modulus is unique and non-repeated, and the maximum singular value is repeated. An inspection of the left and right singular vectors reveals that $\mathbf{x}_1 \neq e^{j\theta}\mathbf{y}_1$ and $\mathbf{x}_2 \neq e^{j\theta}\mathbf{y}_2$ which appears to contradict the MPDA property as stated by both Theorem 1 and Theorem 2. However, note that Theorem 2 states that there must exist at least one major input/output principal direction pair that is aligned. The apparent contradiction can be resolved by realizing that (11) is only one possible orthonormal eigenbasis of $\mathbf{A}^*\mathbf{A}$ whose vectors are right singular vectors. Different orthonormal eigenbasis of $\mathbf{A}^*\mathbf{A}$ are achieved through unitary transformations of the orthonormal basis of the eigenspaces of $\mathbf{A}^*\mathbf{A}$ associated with each particular singular value. The eigenspace of $\mathbf{A}^*\mathbf{A}$ associated with a non-repeating singular value is of rank one; therefore, an orthonormal basis consists of only one vector and all the unitary transformations of this basis are of the scalar form $e^{j\theta}$. On the other hand, the eigenspace of $\mathbf{A}^*\mathbf{A}$ associated with a repeating singular value has rank greater than one, and therefore an orthonormal basis consists of more than one vector, and hence appropriate unitary transformations for this basis take the form of a unitary matrix whose size is given by the rank of the corresponding eigenspace.

Hence, for this example, there must exist a unitary matrix that transforms the left singular vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ into $\mathbf{x}_1'$ and $\mathbf{x}_2'$ such that at least one of the transformed left singular vectors is aligned with the corresponding right singular vectors $\mathbf{y}_1'$ and $\mathbf{y}_2'$. The problem becomes that of finding a matrix $\mathbf{U}$ that satisfy the equations

$$[\mathbf{x}_1' \quad \mathbf{x}_2'] = [\mathbf{x}_1 \quad \mathbf{x}_2]\mathbf{U} \tag{12}$$

$$[\mathbf{y}_1' \quad \mathbf{y}_2'] = [\mathbf{y}_1 \quad \mathbf{y}_2]\mathbf{U} \tag{13}$$

$$\mathbf{x}_1' = e^{j\theta}\mathbf{y}_1'$$

under the unitary constraint

$$\mathbf{U}^*\mathbf{U} = \mathbf{I}$$

The solution can be found by solving an equivalent system of equations that equates the moduli of the elements of $\mathbf{x}_1'$ and $\mathbf{y}_1'$ and that constrains the arguments of elements of $\mathbf{x}_1'$ and $\mathbf{y}_1'$ to differ by θ. Although this is a simple problem in complex algebra, the resulting set of equations have many terms and are relatively cumbersome to manipulate; hence such an approach is not pursued here and an alternative method is used to solve the problem. First, from Lemma 2 it follows that the right eigenvector $\mathbf{v}_1$ is also a right singular vector $\mathbf{y}_1'$. Let $\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2]$ then the first part of the problem becomes finding a normalized vector $\mathbf{u}_1$ such that

$$\mathbf{y}_1' = \mathbf{v}_1 = [\mathbf{y}_1 \quad \mathbf{y}_2]\mathbf{u}_1 \tag{14}$$

The normalized least squares solution to (14) is

$$\mathbf{u}_1 = \frac{[\mathbf{y}_1 \quad \mathbf{y}_2]^+ \mathbf{v}_1}{\left\| [\mathbf{y}_1 \quad \mathbf{y}_2]^+ \mathbf{v}_1 \right\|} = \begin{bmatrix} 0.5548 + 0.8057i \\ 0.2072 + 0.0126i \end{bmatrix}.$$

where $[\bullet]^+$ denotes the Moore-Penrose pseudo-inverse [6]. The second and last part of the problem is to choose $\mathbf{u}_2$ such that $\mathbf{U}$ is unitary. One choice is

$$\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 0.5548 + 0.8057i & 0.2075 \\ 0.2072 + 0.0126i & -0.6026 + 0.7705i \end{bmatrix}$$

Now using the relationships (12) and (13) and defining $\mathbf{X}' = [\mathbf{x}_1' \ \ \mathbf{x}_2' \ \ \mathbf{x}_3]$ and $\mathbf{Y}' = [\mathbf{y}_1' \ \ \mathbf{y}_2' \ \ \mathbf{y}_3]$ yields an alternative singular value decomposition $\mathbf{A} = \mathbf{X}'\Sigma\mathbf{Y}'^*$ where

$$\mathbf{x}_1' = \begin{bmatrix} 0.0088 + 0.1345i \\ -0.5964 + 0.6725i \\ 0.4038 - 0.1041i \end{bmatrix}, \quad \mathbf{x}_2' = \begin{bmatrix} -0.5540 + 0.0969i \\ 0.3042 - 0.2021i \\ 0.7232 - 0.1649i \end{bmatrix}$$

$$\mathbf{x}_3' = \begin{bmatrix} 0.4426 + 0.6853i \\ 0.2421 - 0.0061i \\ 0.2491 + 0.4624i \end{bmatrix}$$

$$\mathbf{y}_1' = \begin{bmatrix} -0.0687 + 0.1159i \\ -0.8719 + 0.2183i \\ 0.3920 + 0.1421i \end{bmatrix}, \quad \mathbf{y}_2' = \begin{bmatrix} 0.4831 - 0.5764i \\ 0.0549 + 0.2386i \\ 0.0228 + 0.6114i \end{bmatrix}$$

$$\mathbf{y}_3' = \begin{bmatrix} -0.6451 \\ -0.0400 + 0.3612i \\ -0.5455 + 0.3927i \end{bmatrix}$$

where the singular values again are

$$\{\sigma_1, \sigma_2, \sigma_3\} = \{2, 2, 1\}$$

Finally, the apparent contradiction of Theorem 2 is dismissed by verifying

$$\mathbf{x}_1' = e^{j\arg(\lambda_1)}\mathbf{y}_1' = e^{-0.6000j}\mathbf{y}_1'$$

Note that $\mathbf{x}_2' \neq e^{j\theta}\mathbf{y}_2'$ even though σ_2 is equal to the maximum singular value. A reasonable question now is whether it is possible to choose $\mathbf{u}_2$ such that $\mathbf{x}_2'$ is also aligned with $\mathbf{y}_2'$. The answer is no. This can be proven by contradiction as follows. By construction $\mathbf{y}_1'$ is an eigenvector of $\mathbf{A}$ corresponding to an eigenvalue of maximum modulus, namely $\mathbf{v}_1$. Next, it can be shown (see Theorem 2, proof of sufficiency) that alignment of a major-input with a major-output principal direction implies that the major principal directions are both necessarily eigenvectors of $\mathbf{A}$. Now, assume that is possible to choose $\mathbf{u}_2$ such that $\mathbf{x}_2'$ and $\mathbf{y}_2'$ are aligned. Necessarily, $\mathbf{y}_2'$ is also an eigenvector of $\mathbf{A}$, which implies that there are two linearly independent eigenvectors, $\mathbf{y}_1'$ and $\mathbf{y}_2'$, associated with the eigenvalue of maximum modulus. Hence, the eigenvalue's geometric multiplicity is greater than one. It is a fact that the geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity. A contradiction then arises because in this case the algebraic multiplicity of the eigenvalue is equal to one, thus implying that the assumption is false.

The result that it is not possible to achieve alignment of all the major input and major output principal directions is not clear from the original statement of the MPDA property as given in Section 3. However, this result is explicitly provided for by the revised version of the MPDA property of Section 4, which allows for the non-alignment of several major input/output principal direction pairs as long as at least one other input/output principal direction pair is aligned, as is the case in this example.

6. Conclusions

This work clarifies and refines the MPDA principle by explicitly considering the case of repeated maximum singular values. An alternative proof of the necessity of the MPDA property is presented, based on dual norm and vector theory. This proof establishes ties between the MPDA property and earlier duality work which partly inspired it. An example shows that the alignment properties of the input/output principal direction pairs associated with a maximum singular value are directly related to the eigenvectors associated with eigenvalues of maximum modulus in terms of both the multiplicity and the extent of alignment.

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