# Robustness of Classical Tuning Correlations for Proportional-Integral Controllers

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## Abstract

*A formal robustness stability analysis of popular proportional-integral (PI) controller tuning rules for systems approximated by a first-order-plus-time-delay models is proposed. The uncertainty in the process model is represented by multiplicative parametric perturbations in the process gain, process time constant, and process time-delay. The Zero-Exclusion Principle is used to characterize the robustness of the uncertain system in terms of the set of all perturbations that result in stable closed-loops. The robustness results recover the standard gain and phase margin concepts as special cases. In addition, a parametric stability margin is introduced for this class of problems as a generic metric via which alternative PI controller tuning rules may be compared in terms of robustness to simultaneous variations in the all three model parameters. The results of the paper can be applied to several disturbance-rejection and tracking PI tuning rules in widespread use, and permits comparing the tuning rules in terms of their relative robustness. It is shown for example that the Integral-Square-Error tuning rule for disturbance rejection can be destabilized by a 7% simultaneous variation in the system parameters.*

## 1. Introduction

The classical proportional-integral-derivative controllers in use since the early 40s are now all-pervasive in industrial applications. In particular, the proportional-integral (PI) version has found widespread application in processes where the presence of measurement noise does not permit taking advantage of the beneficial effects of the derivative action. Since PI controllers are tuned using a number of classical correlations, stemming back to the famous Ziegler-Nichols methods first proposed in 1942 [1], it is therefore relevant to analyze the robust-stability of such tuning correlations. This robust-stability problem has not been extensively studied in the literature, but recent advances in the theory of robust control analysis, in particular dealing with real parametric uncertainties, now make it possible to address the issue in a quantitative fashion.

Since the appearance of Ziegler and Nichols' seminal work , a large body of literature emerged on alternative approaches for tuning controllers of the PI type under the assumption that the process being controlled is adequately described by the stable first-order-plus-delay model

$$p(s) = \frac{Ke^{-\theta s}}{\tau s + 1} \qquad (1)$$

where $K$ is the process gain, $\tau > 0$ is the time constant, and $\theta > 0$ is the time delay. The three model parameters can be obtained from open-loop step-response tests, from statistical parameter-estimation methods, model reduction techniques, *etc.* [2], [3].

Among the large number of correlations in use today based on the first-order-plus-dead-time process paradigm and found in most introductory texts on control engineering are the widely-cited prescriptions proposed by Cohen and Coon [4], the correlations based on error-integral criteria proposed and developed in [5], [6], [7], and [8] and that include performance criteria such as the integral of the absolute value of the error (IAE), the integral of the squared error (ISE), and the integral of the time-averaged absolute error (ITAE). A summary of relevant correlations is given in Table I. All of the tuning correlations mentioned have been carefully developed by their authors to approximately satisfy specific performance criteria, and to ensure that the closed loop is stable. More recent work [9] [10] [11] [12] is not further explored in this paper, because the earlier contributions are more often cited and are therefore more reasonable candidates for the robust analysis technique presented.

Although in all cases the tuning correlations recognize that model (1) is an approximation, they do not address the fact that the process of parameter identification is inherently affected by uncertainties. More specifically, the gain, time-constant, and time-delay parameters may be respectively affected by errors $\Delta K$, $\Delta\tau$, and $\Delta\theta$. This paper seeks to quantify the robust stability of classical tuning correlations for PI controllers with respect to uncertainties in the gain, time-constant, and the time-delay parameters of the model. A major objective of this paper is the calculation of a meaningful parametric stability margin for this class of systems, which can then be used as a quantitative metric to compare alternative PI controller tuning rules with respect to robust stability.

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The proofs of the various theorems and lemmas are omitted for brevity; however, complete details are available in [14].

# 2. Preliminaries

The uncertain process and the PI controller are arranged in the standard feedback configuration shown in Figure 1, where the variable $y(s)$ is the process output, $r(s)$ is the set point, $d(s)$ is an additive disturbance, and $e(s) = r(s) - y(s)$ is the feedback error.

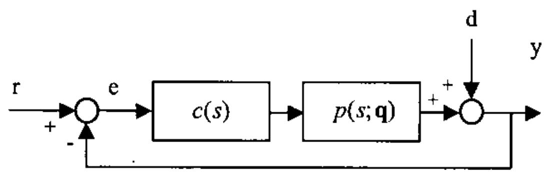


Figure 1. Feedback control structure with proportional-integral controller $c(s)$ and uncertain process $p(s;\, \mathbf{q})$.

### 2.1. Process model and uncertainty description

For the purposes of this study, it is convenient to represent the uncertain first-order plus delay model as

$$p(s;\mathbf{q}) = \frac{Ke^{-\theta s}}{\tau s+1} \qquad (2)$$

where the process gain $K$, the time constant $\tau > 0$, and is the time delay $\theta > 0$ are real parameters, and where the uncertainty in the parameters is expressed in the multiplicative form

$$K = \alpha_K K_0\,,\ \tau = \alpha_\tau \tau_0\ ,\ \theta = \alpha_\theta \theta_0 \qquad (3)$$

where $K_0 \neq 0$, $\tau_0 > 0$, and $\theta_0 > 0$ are the known nominal values of the process parameters. The scalars $\alpha_K > 0$, $\alpha_\tau > 0$, and $\alpha_\theta > 0$ are unknown real multiplicative perturbations that are collected in the uncertainty vector $\mathbf{q} = [\alpha_K, \alpha_\tau, \alpha_\theta]^T \in Q \subset \Re^3$, which belongs to the uncertainty domain $Q$ composed of vectors with strictly positive elements. The nominal process $p_0(s) := p(s;\mathbf{q}_0)$ is recovered from (2) after setting $\mathbf{q} = \mathbf{q}_0 := [1\ 1\ 1]^T$ to yield

$$p_0(s) = \frac{K_0 e^{-\theta_0 s}}{\tau_0 s+1} \qquad (4)$$

Note that the real multiplicative parametric uncertainty description (3) has the associated additive perturbations

$$\Delta K := K_0(\alpha_K - 1) \qquad (5a)$$

$$\Delta\tau := \tau_0(\alpha_\tau - 1) \qquad (5b)$$

$$\Delta\theta := \theta_0(\alpha_\theta - 1) \qquad (5c)$$

so that $K = K_0 + \Delta K$, $\tau = \tau_0 + \Delta\tau$, and $\theta = \theta_0 + \Delta\theta$. Also, a scaled additive perturbation is defined as

$$\Delta\mathbf{q} := \left[\frac{\Delta K}{K_0}\ \ \frac{\Delta\tau}{\tau_0}\ \ \frac{\Delta\theta}{\theta_0}\right]^T$$

such that $\mathbf{q} = \mathbf{q}_0 + \Delta\mathbf{q}$. The strict positivity of the multiplicative parameter $\alpha_K$ in (3) implies that the sign of the gain $K$ is invariant in $Q$, a constraint that is normally met in applications since often the sign of the gain is known from the physics of the underlying problem or from experience.

### 2.2. Proportional-integral control and controller tuning rules

The proportional-integral controller considered in Figure 1 is of the classical form

$$c(s) = K_c\left(1 + \frac{1}{\tau_I s}\right) \qquad (6)$$

where the controller gain $K_c \neq 0$ and the integral time-constant $\tau_I > 0$ are adjustable parameters. Some of the historically and practically most important tuning correlations are given in Table 1, where the prescribed values of control gain and integral time-constant are expressed through equations that require knowledge of the known nominal process parameters $K_0$, $\tau_0$, and $\theta_0$ of the nominal model (4). Note that the entries in the table depend on the time-delay-to-time-constant ratio $\theta_0/\tau_0$, a fact that is exploited later in the paper. These tuning correlations have been developed and tested for nominal models in the range $0.1 \le \theta_0/\tau_0 \le 1.0$, which represents a very wide range of practical processes of interest. After invoking any of these tuning correlations, the parameters of the PI controller (6) could be written in the form $K_c = K_c(K_0, \theta_0/\tau_0)$ and $\tau_I = \tau_I(\tau_0, \theta_0/\tau_0)$ where the specific functional dependencies are given by the table entries, and hence it follows that $c(s) = c(s;\mathbf{q}_0)$.

# 3. Analysis of robust stability

### 3.1. Conditions for robust stability

Straightforward block-diagram algebra operations show that the stability of the closed-loop system of Figure 1 is determined by the properties of the quasipolynomial of degree 2 [13]

$$\delta(s;\mathbf{q}) = \alpha_K K_0 K_c(\tau_I s+1)e^{-\alpha_\theta\theta_0 s} + \alpha_\tau\tau_0\tau_I s^2 + \tau_I s \ (7)$$

where $\mathbf{q} \in Q$, and where the PI control parameters $K_c$ and $\tau_I$ are selected using a tuning correlation taken from Table 1 and are therefore functions of $\mathbf{q}_0$. It is known that the *nominal quasipolynomial* $\delta(s;\mathbf{q}_0)$ is Hurwitz, because the tuning correlations considered in Table 1 yield controller settings that guarantee the stability of the nominal closed loop. Let

$$\Delta(s) := \{\delta(s;\mathbf{q}) \mid \mathbf{q} \in Q\} \qquad (8)$$

denote the family of quasipolynomials generated by (7) for all $\mathbf{q} \in Q$.

In the suite the symbol $\tilde{\omega}$ is used to denote the standard frequency variable measured in reciprocal seconds, while the symbol $\omega = \tilde{\omega}\theta_0$ is used to denote a dimensionless frequency. The *robust stability* of the closed-loop with respect to an uncertainty domain $Q$ is ensured if and only if the entire family of quasipolynomials $\Delta(s)$ is Hurwitz. This in turn can be ensured through the Zero-Exclusion Principle enunciated in the following Lemma, which follows from the application of the Boundary Crossing Theorem [13] for quasipolynomials,

**Lemma 1.** *Given the parametric uncertainty* $\mathbf{q} \in Q$ *and the family of quasipolynomials* $\Delta(s)$ *of constant degree whose nominal quasipolynomial* $\delta(s;\mathbf{q}_0)$ *is Hurwitz, then every element of* $\Delta(s)$ *is Hurwitz if and only if the image set* $\Delta(j\tilde{\omega})$ *excludes the origin for all* $\tilde{\omega} \geq 0$.

Applying Lemma 1 to the system being considered yields the following stability result.

**Lemma 2.** *The closed-loop system of Figure 1 with PI control parameters adjusted using a tuning correlation from Table 1 is robustly stable with respect to all parametric uncertainties* $\mathbf{q} = [\alpha_K\ \alpha_\tau\ \alpha_\theta]^T \in Q$ *if and only if the inequality*

$$\mathbf{A}(\omega,\alpha_\theta)\begin{bmatrix}\alpha_K \\ \alpha_\tau\end{bmatrix} \neq \mathbf{b}(\omega) \qquad (9a)$$

*holds for all* $\mathbf{q} \in Q$ *and for all* $\omega \geq 0$, *where*

$$\mathbf{b}(\omega) := \begin{bmatrix} 0 \\ -\omega \end{bmatrix} \in \Re^2 \qquad (9b)$$

and where $\mathbf{A}(\omega,\alpha_\theta) = \begin{bmatrix} a_{11} & a_{21} \\ a_{21} & 0 \end{bmatrix}$ contains the elements

$$a_{11} = K_0K_c\left(\frac{\theta_0}{\tau_0}\left(\frac{\tau_0}{\tau_I}\right)\cos(\alpha_\theta\omega) + \omega\sin(\alpha_\theta\omega)\right)$$

$$a_{21} = -\left(\frac{\theta_0}{\tau_0}\right)^{-1}\omega$$

$$a_{21} = K_0K_c\left(\omega\cos(\alpha_\theta\omega) - \frac{\theta_0}{\tau_0}\left(\frac{\tau_0}{\tau_I}\right)\sin(\alpha_\theta\omega)\right)$$

The net effect of using multiplicative perturbations and the dimensionless frequency $\omega$ as opposed to the standard frequency $\tilde{\omega}$ is that the expressions in (9) explicitly shows the role of the ratio $\theta_0/\tau_0$ needed to extract tuning parameters from Table 1; hence, the product $(\tau_0/\tau_I)(\theta_0/\tau_0)$ is not further simplified. Furthermore, Lemma 2 completely describes the robust stability characteristics of the system.

### 3.2. Parametric boundaries for robust stability

The theoretical development of the previous section is valid for any arbitrary uncertainty domain $Q$ that is a simply-connected *open* subspace of $\Re^3$ consisting of strictly positive elements and that contains the nominal point $\mathbf{q}_0 = [1,1,1]$. This section seeks to characterize the largest uncertainty domain for which the closed-loop system is robustly stable, thereby giving the region of all stabilizable multiplicative perturbations of the nominal parameters. Such a region, denoted $Q_{\max}$, is fully described by its boundary $\partial Q_{\max}$. Obviously, any vector $\mathbf{q}$ for which $\alpha_K = 0$, $\alpha_\tau = 0$, or $\alpha_\theta = 0$ is a possible element of the boundary of $Q_{\max}$. The challenge is to find all the strictly positive vectors $\mathbf{q}$ that are elements of $\partial Q_{\max}$. First note that for every strictly positive element $\overline{\mathbf{q}} = [\overline{\alpha}_K\ \overline{\alpha}_\tau\ \overline{\alpha}_\theta]^T$ of the parametric robust-stability boundary, the characteristic quasi-polynomial $\delta(s;\overline{\mathbf{q}})$ is not Hurwitz and the family of quasipolynomials produced by $Q = \{\overline{\mathbf{q}} + \Delta\mathbf{q}, \|\Delta\mathbf{q}\| < \varepsilon\}$ contains at least one element that is Hurwitz, where $\varepsilon$ is an arbitrary positive real scalar and $\|\bullet\|$ is a vector norm. This fact gives rise to the following theorem.

**Theorem 1.** *If the strictly positive uncertainty vector* $\mathbf{q} = [\alpha_K\ \alpha_\tau\ \alpha_\theta]^T$ *is an element of the parametric robust-stability boundary* $\partial Q_{\max}$ *for the closed-loop system of Figure 1 with PI control parameters adjusted using a tuning correlation from Table 1, then for some* $\omega \geq 0$ *the elements of vector* $\mathbf{q}$ *must satisfy the parametrized map*

$$\alpha_K = \frac{-\omega}{K_0K_c\left(\omega\cos(\alpha_\theta\omega) - \frac{\theta_0}{\tau_0}\left(\frac{\tau_0}{\tau_I}\right)\sin(\alpha_\theta\omega)\right)} \qquad (10a)$$

$$\alpha_\tau = \frac{-\left(\frac{\theta_0}{\tau_0}\left(\frac{\tau_0}{\tau_I}\right)\cos(\alpha_\theta\omega) + \omega\sin(\alpha_\theta\omega)\right)\frac{\theta_0}{\tau_0}}{\left(\omega\cos(\alpha_\theta\omega) - \frac{\theta_0}{\tau_0}\left(\frac{\tau_0}{\tau_I}\right)\sin(\alpha_\theta\omega)\right)\omega} \qquad (10b)$$

Note that for a given ratio $\theta_0/\tau_0$ the candidate uncertainty-boundary coordinates $\alpha_K$ and $\alpha_\tau$ in (10) are parametrized in terms of $\alpha_\theta$. Equations (10a)-(10b) trace curves in the $\alpha_\tau - \alpha_K$ space as the frequency varies for a fixed value of $\theta_0/\tau_0$ and for an arbitrarily selected value $\alpha_\theta > 0$. A subset of these curves defines the boundary set $\partial Q_{\max}$. In particular, note that if at a given frequency the values $\alpha_K$ and $\alpha_\tau$ obtained from (10) are not simultaneously positive, then they are not elements of $\partial Q_{\max}$ because such pairs are not admissible. Therefore, it

Table 1. Controller tuning correlations for proportional-integral.

| Method | Problem | $K_0K_c$ | $\tau_0/\tau_I$ | Reference |
|---|---|---|---|---|
| Ziegler-Nichols | Servo control | $0.9(\theta_0/\tau_0)^{-1}$ | $0.3(\theta_0/\tau_0)^{-1}$ | [1] |
| Cohen-Coon | Servo control | $\frac{1}{12}+0.9(\theta_0/\tau_0)^{-1}$ | $\frac{20+9(\theta_0/\tau_0)^{-1}}{30+3(\theta_0/\tau_0)}$ | [4] |
| IAE | Servo control | $0.758(\theta_0/\tau_0)^{-0.861}$ | $1.02-0.323(\theta_0/\tau_0)$ | [7] |
| ITAE | Servo control | $0.586(\theta_0/\tau_0)^{-0.916}$ | $1.03-0.165(\theta_0/\tau_0)$ | [7] |
| ISE | Regulation control | $1.305(\theta_0/\tau_0)^{-0.960}$ | $0.492(\theta_0/\tau_0)^{-0.739}$ | [5] |
| IAE | Regulation control | $0.984(\theta_0/\tau_0)^{-0.986}$ | $0.608(\theta_0/\tau_0)^{-0.707}$ | [5] |
| ITAE | Regulation control | $0.859(\theta_0/\tau_0)^{-0.977}$ | $0.674(\theta_0/\tau_0)^{-0.680}$ | [5] |

can be anticipated that $\partial Q_{\max}$ can be characterized by evaluating (10) within a set of selected frequency intervals, since at some frequencies the maps yield inadmissible solutions. The subintervals of frequency that yield admissible solutions can be identified by finding the frequencies at which $\alpha_K$ and $\alpha_\tau$ defined in (10) are simultaneously positive.

Consider the following definitions. First, let $\omega_{n1}$ be the smallest positive zero of

$$f_n(\omega):=\frac{\tau_0}{\tau_I}\frac{\theta_0}{\tau_0}\cos(\alpha_\theta\omega)+\omega\sin(\alpha_\theta\omega) \qquad (11)$$

and let $\omega_{d1}<\omega_{d2}$ be the two smallest positive zeros of

$$f_d(\omega):=\omega\cos(\alpha_\theta\omega)-\frac{\tau_0}{\tau_I}\frac{\theta_0}{\tau_0}\sin(\alpha_\theta\omega) \qquad (12)$$

**Theorem 2**. *The strictly positive uncertainty vector* $\overline{\mathbf{q}}=[\overline{\alpha}_K\,\overline{\alpha}_\tau\,\overline{\alpha}_\theta]^T$ *is an element of the parametric robust-stability boundary* $\partial Q_{\max}$ *for the closed-loop system of Figure 1 with PI control parameters adjusted using a tuning correlation from Table 1 if and only if* $\overline{\mathbf{q}}$ *satisfies* (10) *for some* $\omega\in\Omega_1$, *where*

$$\Omega_1:=\begin{cases}(0,\omega_{n1}) & \text{if } \quad 1\le\frac{\tau_0}{\tau_I}\frac{\theta_0}{\tau_0}\alpha_\theta<\infty\\ (\omega_{d1},\omega_{n1}) & \text{otherwise}\end{cases}$$

Theorem 2 defines the region of stable parameter perturbations $Q_{\max}$ in terms of its boundary for a fixed perturbation in time delay, $\alpha_\theta$. For the nominal value $\alpha_\theta=1$, the above results give the stability boundary when there is no uncertainty in the time delay. To characterize the effect of arbitrary perturbations in the time delay, the above analysis can be performed over a range of values $\alpha_\theta>0$. The result is a different stability-boundary curve for each value of $\alpha_\theta$, and hence the stability boundary becomes a surface in the $\alpha_\tau-\alpha_\theta-\alpha_K$ space. Hence, every point $\overline{\mathbf{q}}=[\overline{\alpha}_K\,\overline{\alpha}_\tau\,\overline{\alpha}_\theta]^T$ on the stability boundary surface must satisfy

$$\overline{\alpha}_K=f(\overline{\alpha}_\tau,\overline{\alpha}_\theta) \qquad (13)$$

where $f(\overline{\alpha}_\tau,\overline{\alpha}_\theta)$ depends implicitly on the tuning rule adopted from Table 1 because it features the product $K_0K_c$ and the ratio $\tau_0/\tau_I$. Given a pair of boundary coordinates $\overline{\alpha}_\tau$ and $\overline{\alpha}_\theta$, the mapping (13) if found as follows. First, for notational simplicity let $\alpha_K=f_K(\omega,\alpha_\theta)$ and $\alpha_\tau=f_\tau(\omega,\alpha_\theta)$ respectively represent equations (10a) and (10b) whose right-hand sides unambiguously define the mappings $f_K:(\omega,\alpha_\theta)\to\alpha_K$ and $f_\tau:(\omega,\alpha_\theta)\to\alpha_\tau$. Note that for a given coordinate $\alpha_\theta$ the mapping $f_\tau$ defines a one-to-one correspondence between $\omega\in\Omega_1$ and $\alpha_\tau\le 0$. This implies that the inverse map $f_\tau^{-1}:(\alpha_\tau,\alpha_\theta)\to\omega$ exists. Next, set $\alpha_\theta=\overline{\alpha}_\theta$ and $\alpha_\tau=\overline{\alpha}_\tau$ in (10b) to define $\overline{\alpha}_\tau=f_\tau(\omega,\overline{\alpha}_\theta)$, and solve the equation to obtain $\omega=f_\tau^{-1}(\overline{\alpha}_\tau,\overline{\alpha}_\theta)$ with $\omega\in\Omega_1$ as prescribed by Theorem 2. Finally, substitute the frequency value into (10a) to obtain

$$\overline{\alpha}_K=f_K(f_\tau^{-1}(\overline{\alpha}_\tau,\overline{\alpha}_\theta),\overline{\alpha}_\theta):=f(\overline{\alpha}_\tau,\overline{\alpha}_\theta)$$

Hence, the map (13) is readily computable. It may be of interest to remark that in general $f(\overline{\alpha}_\tau,\overline{\alpha}_\theta)$ does not have a closed-form expression, because $f_\tau^{-1}(\bar{\alpha}_\tau, \bar{\alpha}_\theta)$ itself does not have a closed-form expression. Extensive numerical studies have shown, however, that approximate closed-form expressions for $f_\tau^{-1}$ can be obtained with high accuracy through least-squares fit to simple functional forms [14]. Finally, the system is stable with respect to the arbitrary multiplicative uncertainty $\mathbf{q} = [\alpha_K \; \alpha_\tau \; \alpha_\theta]^T$ if the uncertainty satisfies

$$\alpha_K < f(\alpha_\tau, \alpha_\theta) \qquad (14)$$

Using this expression it can easily be determined if a given set of uncertainties destabilizes the system for a specific tuning rule. Furthermore, it follows that a complete characterization of the region of stabilizable multiplicative perturbations is compactly given by

$$Q_{\max} := \{\mathbf{q} = [\alpha_K \; \alpha_\tau \; \alpha_\theta]^T : \quad 0 < \alpha_K < f(\alpha_\tau, \alpha_\theta), \; \alpha_\tau > 0, \alpha_\theta > 0\}$$

Clearly, the *size* of $Q_{\max}$ can be used to compare different tuning rules. Obviously, tuning rules with larger regions of stable parameter perturbations are more robustly stable. The next section introduces an appropriate parametric stability margin to better characterize the size of the stability region. The parametric stability margin is used to characterize the robustness of the controllers for quantitative purposes when considering a given tuning rule, and for qualitative purposes when comparing alternative tuning rules.

### 3.3. Stability Margins

The parametric stability margin is defined as the length (in a vector-norm sense) of the smallest scaled additive perturbation $\Delta\mathbf{q} = \mathbf{q} - \mathbf{q}_0$ that destabilizes the closed-loop system of Figure 1. This margin serves as a quantitative measure of the robustness of the closed system with respect to the parametric uncertainty referenced to the nominal point $\mathbf{q}_0 = [1,1,1]^T$, and is useful as a means for comparing the performance of proposed controller tuning rules. The value of the parametric stability margin depends on the norm used to measure the length of the smallest destabilizing perturbation $\Delta\mathbf{q}$. In this paper the $l_\infty$ norm is adopted because it represents the smallest *box* that destabilizes the system and that contains the nominal point in the uncertain parameter space. Other norms may certainly be considered (for example the $l_2$ represents a sphere in the uncertain parameter space), but the $l_\infty$ is chosen because the resulting box gives an easily understood bound in terms of the physical parameters of the system.

The $l_\infty$ parametric stability margin is calculated by solving the minimization problem

$$\rho_{K\tau\theta}(\theta_0/\tau_0) := \min_{\alpha_\theta > 0, \omega \in \Omega_1} \left\| [\alpha_K - 1, \alpha_\tau - 1, \alpha_\theta - 1]^T \right\|_\infty \qquad (15)$$

where $\Omega_1$ is the interval of frequencies that gives the robust stability region as prescribed by Theorem 2, and $\alpha_K$ and $\alpha_\tau$ are given by (10) in Theorem 1. Note that (15) explicitly shows the dependence on the ratio $\theta_0/\tau_0$, the only free parameter when considering a specific tuning rule chosen from Table 1. This optimization has no convexity guarantees, but because it only involves two parameters, it can easily be solved via an exhaustive numerical search. In addition, the constraint that $\alpha_\theta > 0$ can be further restricted. To show this, for $\alpha_\theta = 1$ (*i.e.*, when there is no perturbation in the time delay) let

$$\rho_{K\tau}(\theta_0/\tau_0) := \min_{\omega \in \Omega_1} \left\| [\alpha_K - 1, \alpha_\tau - 1]^T \right\|_\infty$$

be the parametric stability margin when considering only uncertainty in the process gain and in the process time constant. This margin is easily calculated by performing a frequency sweep over the specified range. The minimization (15) need only be performed over the range $1 - \rho_{K\tau}(\theta_0/\tau_0) \le \alpha_\theta \le 1 + \rho_{K\tau}(\theta_0/\tau_0)$, because $\rho_{K\tau}(\theta_0/\tau_0)$ is an upper bound, and for every $\alpha_\theta$ outside this range

$$\left\| [\alpha_K - 1, \alpha_\tau - 1, \alpha_\theta - 1]^T \right\|_\infty \ge \rho_{K\tau}(\theta_0/\tau_0).$$

The parametric stability margin (15) considers simultaneous perturbations in all the uncertain parameters, and hence is a more meaningful measure of robustness than the classical gain and phase margin [2] metrics.

## 4. Results of Numerical Studies

Using the theoretical development of the previous section it is possible to determine the parametric stability margin $\rho_{K\tau\theta}(\theta_0/\tau_0)$ given in (15) for various values of the ratio $\theta_0/\tau_0$ for all the tuning rules shown in Table 1. The results of this numerical analysis are plotted in Figure 2. For higher values of $\theta_0/\tau_0$, the figure shows that the order of increasing robustness for the tuning rules is ISE-reg., CC, IAE-reg., ITAE-reg., ZN, IAE-servo, and ITAE-servo. For lower values of $\theta_0/\tau_0$, the CC and IAE-reg. tuning rules change their order of robustness. The ZN and ITAE-reg. tuning rules also change their order of robustness for lower values of $\theta_0/\tau_0$. The figure is interpreted as follows. For $\theta_0/\tau_0 = 0.1$ the ISE-reg. parametric stability margin is $\rho_{K\tau\theta} = 0.07$. This implies that a 7% additive perturbation of the nominal process parameters causes an unstable closed-loop system. In fact, for any values of $K_0$, $\tau_0$, and a value $\theta_0$ satisfying $\theta_0/\tau_0 = 0.1$, a controller designed using the ISE regulation tuning rules is destabilized by the process values $K = 1.07K_0$, $\tau = 0.93\tau_0$, and $\theta = 1.07\theta_0$. The ISE-reg. tuning rule is not very robust in the sense that the nominal values of $K_0$, $\tau_0$, and $\theta_0$ are only

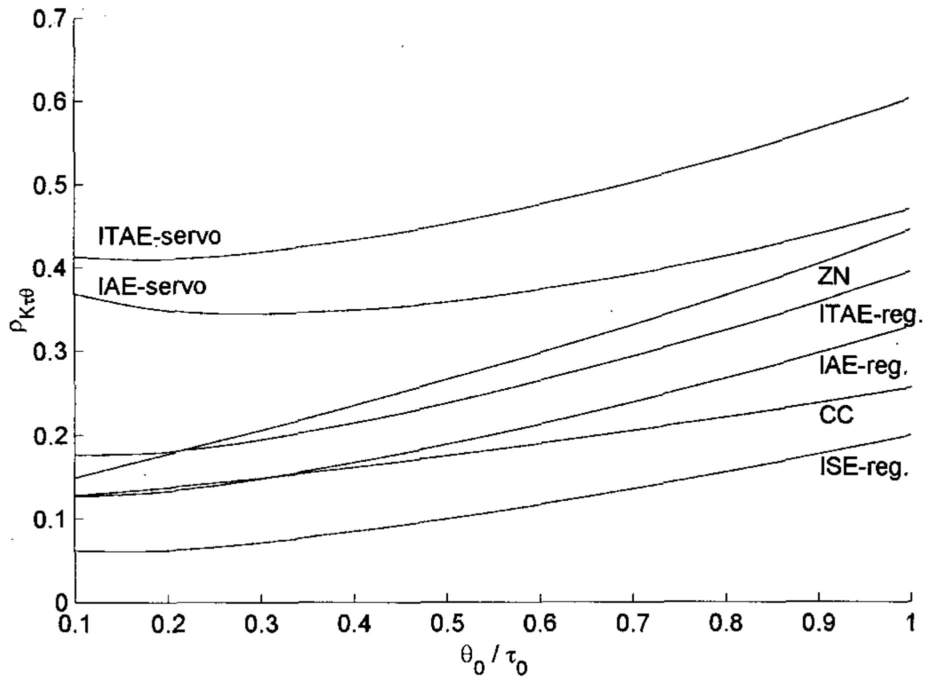


Figure 2. Parametric stability margin of the tuning rules *vs*. the tuning parameter.

estimates for which the approximation error could easily be greater that 0.07. As for the robustness of the other tuning rules, the control engineer could take advantage of available estimates of modeling error bounds and use Figure 7 as a guide in choosing a sufficiently robust controller tuning rule.

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