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A New Perspective on Computing Robust Stability Margins for Complex Parametric Uncertainties

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Abstract

A new and simple approach to the problem of finding robust stability margins for SISO systems with complex parametric uncertainties is proposed. The technique is based on the recently developed critical direction theory, and is an alternative to existing methods such as those based structured singular value formulations. The uncertainties considered belong to highly structured domains satisfying a radial convexity condition. Explicit and exact stability conditions are derived using intuitive geometric arguments. The approach recovers previous results given in the literature for the special case of disk-bounded affine complex uncertainties. Examples are given for rectangular and elliptical uncertainty domains.

1. Introduction

Considerable attention has been given to the problem of assessing robust stability and computing stability margins for SISO systems with parametric uncertainties. It is now well known that affine parametric uncertainty problems result in a rank-one μ formulation. While the general robust stability margin computation problem is known to be NP-hard [3], its rank-one counterpart results in a convex optimization problem that can be readily computed, and can be often derived as an explicit analytical expression [1]-[2] [4]-[6]. This note focuses on the complex parametric uncertainty problem from the perspective of the recently proposed *critical direction* paradigm [7]. In this paper we consider complex parametric uncertainties where each individual uncertainty lies in a highly structured domain that is convex along any line from the origin. This is a mild convexity condition, since the actual domain could be highly non-convex. Exact analytical results are derived for geometrically simple uncertainty sets such as ellipses and rectangles.

2. Background and Preliminaries

Consider the frequency response of a SISO open-loop transfer function

$$g(s,q) = \frac{n(j\omega,0) + \sum_{k=1}^{m} n_k(j\omega)q_k}{d(j\omega,0) + \sum_{k=1}^{m} d_k(j\omega)q_k} \tag{1}$$

where $n(j\omega,0)$ and $d(j\omega,0)$ are nominal polynomials, and the uncertainties q_k belong to radially convex domains Q_k. Hence, $q \in Q$, where $Q = Q_1 \times Q_2 \times \times Q_m \subset C^m$. Let ∂Q_k denote the boundary of domain Q_k. It is assumed that the elements $\partial q_k(j\omega) \in \partial Q_k$ are of the radially convex form

$$\partial q_k(j\omega) = \gamma_k(\omega,\theta_k)e^{j\theta_k}, \quad k = 1,2,\cdots,m \tag{2}$$

where $0 \le \theta_k \le 2\pi$, and where $\gamma_k(\omega,\theta_k) = |\partial q_k(j\omega)| \ge 0$ is a frequency- and phase-dependent magnitude.

Under a unity negative-feedback arrangement the characteristic polynomial for system (1) is given by the polynomial sum

$$p(j\omega,q) = p_o(j\omega) + \sum_{k=1}^{m} p_k(j\omega)q_k(j\omega) \tag{3}$$

where

$$p_o(j\omega) := n(j\omega,0) + d(j\omega,0)$$

$$p_k(j\omega) := n_k(j\omega) + d_k(j\omega) = |p_k(j\omega)|e^{j\theta_{p_k}(j\omega)}$$

Finally, define the *Parametric Robust Stability Margin*

$$\alpha^*(\omega) = \min_{\alpha \in \Re^+} \{\alpha \mid 0 \in p(j\omega,\alpha q) \text{ for some } q \in Q\} \tag{4}$$

which is a nonnegative real scalar that can be interpreted as the minimal magnification (or contraction) of the uncertainty set Q that brings the closed-loop system to the edge of stability. Geometrically, $\alpha^*(\omega)$ represents the minimum tolerable blow-up factor.

2.1. Critical-Direction Theory for Polynomials

The critical-direction theory proposed for rational systems in [7] can be readily modified for the case of polynomials through the definitions given below.

(i) The Critical Direction

$$d(j\omega) := -\frac{p_o(j\omega)}{|p_o(j\omega)|} := e^{j\theta_c(\omega)}$$

is the unit vector which defines the direction from the nominal point $p_o(j\omega)$ towards the origin. It is uniquely identified by the critical phase angle $\theta_c(\omega)$.

(ii) The Uncertainty Template (or value set)

$$\mathcal{T}(\omega) := \{ p(j\omega,q) \in C \mid q_k \in Q_k, k = 1,2,\cdots,m \}$$

(iii) The Critical Template

$$\mathcal{T}_c(\omega) := \{ p(j\omega,q) \in \mathcal{T}(\omega) \mid p(j\omega,q) = p_o(j\omega) + r\,d(j\omega), \text{ for some } r \in \Re^+ \}$$

(iv) The Critical Perturbation Radius

$$\rho_c(\omega) := \max_{r \in \Re^+} \{ r \mid z = p_o(j\omega) + r\, d(j\omega) \in \mathcal{T}_c(\omega) \} \quad (5)$$

(v) The Nyquist Robust Stability Margin

$$k_N(\omega) := \frac{\rho_c(\omega)}{|p_o(j\omega)|} \quad (6)$$

3. Main Results

A robust stability lemma is formulated by analogy with the approach in [7], and is then used to derive the main theorem.

Lemma. *Consider the uncertain characteristic polynomial (3) and assume that nominal stability is attained under unity feedback, and that the critical templates $\mathcal{T}_c(\omega)$ are convex at all frequencies. Then the uncertain system remains stable for all parameter uncertainties $\mathbf{q} \in Q$ if and only if $k_N(\omega) < 1 \ \forall \omega$.*

Theorem. *Consider the uncertain system (1) and the radially-convex uncertainty description (2). Then the parametric robust stability margin (4) is given by the explicit analytical expression*

$$\alpha^*(\omega) = \frac{|p_o(j\omega)|}{\rho_c(\omega)} \quad (7a)$$

where

$$\rho_c(\omega) = \sum_{k=1}^{m} |p_k(j\omega)| \ \gamma_k\left(\omega, \theta_c(\omega) - \theta_{p_k}(\omega)\right) \quad (7b)$$

Proof. Equation (7b) follows from a geometric solution to the maximization problem (5). Expression (7a) follows after invoking the limiting stability condition $k_N(\omega)$ of the Lemma, and then using (3) to solve the minimization problem (4). Further details are omitted for brevity.

3.1. Examples

Two specific examples of radially convex uncertainty domains of interest are considered, namely, rectangular and elliptical descriptions.

Rectangular descriptions. Let $A_k(\omega) > 0$ and $B_k(\omega) > 0$ respectively represent the frequency-dependent half-width and the half-height of a rectangle centered at the origin of the complex plane (Fig. 1a). Let $\phi_k(\omega) := \arctan(B_k(\omega)/A_k(\omega))$ and consider the phase sets $\Theta_{ak} := [0,\ \phi_k) \cup [\pi - \phi_k,\ \pi + \phi_k) \cup [2\pi - \phi_k, 2\pi]$ and $\Theta_{bk} := [\phi_k, \pi - \phi_k) \cup [\pi + \phi_k, 2\pi - \phi_k)$. Then an analytical expression for the boundary-point magnitude appearing on (2) is of the form

$$\gamma_k(\omega, \theta_k) = \frac{A_k(\omega)}{|\cos(\theta_k(\omega))|}\delta_{ak} + \frac{B_k(\omega)}{|\sin(\theta_k(\omega))|}(1 - \delta_{ak}) \quad (8)$$

where $\delta_{ak} = 1$ if $\theta_k \in \Theta_{ak}$, otherwise $\delta_{ak} = 0$. Finally, an exact analytical expression for $\alpha^*(\omega)$ is obtained from (7a) after substituting (8) into (7b).

Elliptical descriptions. Let $A_k(\omega) > 0$, $B_k(\omega) > 0$, and $\phi_k(\omega)$ respectively represent the frequency-dependent semi-major axis, semi-minor axis, and the orientation with respect to the real axis of the ellipse centered at the origin of the complex plane (Fig. 1b). Then an analytical expression for the boundary-point magnitude appearing on (2b) is of the form

$$\gamma_k(\omega, \theta_k) = \left[A_k(\omega)^2 \cos^2(\theta_k - \phi_k(\omega)) + B_k(\omega)^2 \sin^2(\theta_k - \phi_k(\omega))\right]^{1/2} \quad (9)$$

Obviously, the case of a circular uncertainty is recovered from (9) after setting $A_k(\omega) = B_k(\omega)$. An exact analytical expression for $\alpha^*(\omega)$ is obtained from (7a) after substituting (9) into (7b). For the circular uncertainty case, the result given in [5] is recovered.

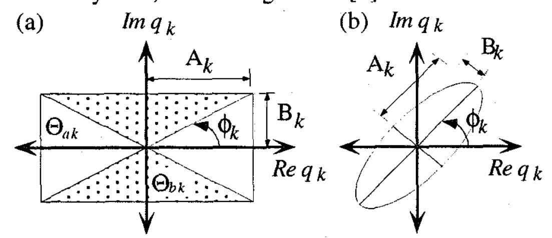

Figure 1. Examples of radially-convex complex uncertainty regions: (a) rectangular and (b) elliptical.

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