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Asymptotic Stability Independent of Delays: Simple Necessary and Sufficient Conditions

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Abstract

In this paper we study stability properties of linear time-invariant delay systems. We consider specifically the notion of asymptotic stability independent of delays. Systems with both commensurate and non-commensurate delays are investigated. We present for each case a necessary and sufficient condition, and further we demonstrate how these conditions may be extended readily to study asymptotic stability independent of delays for uncertain systems. These conditions can be readily checked, and they appear to be considerably simpler than results developed elsewhere previously.

1 Introduction

Stability problems for time delay systems have been a topic of long standing interest; the monographes [9] and [18] record extensive studies on these problems. A specific notion concerning stability of delay systems is *asymptotic stability independent of delays*. This notion was formally introduced in, e.g., [14] and has received considerable research interest in the past decade. Roughly, a time delay system is said to be asymptotically stable independently of delays if that system maintains stability with respect to all nonnegative delay parameters.

The earlier development on asymptotic stability independent of delays have led to many useful results, particularly for linear time invariant systems. Noteworthy is the so-called *two-variable criterion* developed in [13], [14]. This result was initially deemed both necessary and sufficient for asymptotic stability independent of delays, but later was found in [15] to be only sufficient. An alternative criterion was developed in [4] based on the two-variable criterion, which is sufficient as well. Though as noted in [15], these conditions are "close to" and may be modified to being necessary, an additional condition must be imposed. This extra condition may restrict significantly the class of systems which may be considered. More recently, a necessary and sufficient condition was given in [10]. This result is similar to and has essentially the same level of difficulty as the two-variable criterion, which, as noted in [10], is extremely difficult to verify. Additionally, a necessary and sufficient condition of a different nature was reported in [17]; however, this result is valid only for a limited class of delay systems. In contrast, numerous sufficient conditions have been obtained by many researchers (see, e.g., [3], [11], [19], [21], [25]). Though these sufficient conditions may be arbitrarily weaker than those of [14], [4], [10], they are significantly less intensive in computation.

In this paper we consider asymptotic stability independent of delays for linear time invariant systems. Delay systems with both commensurate and non-commensurate delays will be addressed. We develop for each class of systems a necessary and sufficient condition. An appealing advantage offered by these conditions is that they are not only stronger than the criteria of [14], [4], but also are considerably simpler in computation than these criteria and in turn the necessary and sufficient condition given in [10].

Our derivation takes an approach motivated by the currently prevailing research in robust stability analysis. This approach proves both natural and fruitful for studying asymptotic stability independent of delays. Indeed, in an inexact sense, asymptotic stability independent of delays may itself be interpreted as a robust stability problem. Our development leads to several necessary and sufficient conditions that possess the form of typical small gain conditions in robust stability analysis. An important robust stability measure known as the structured singular value [6], [24] is used to quantify these conditions. As such, these conditions may be tested efficiently by using the well-developed computational methods in robust stability analysis. Furthermore, since they are small gain conditions, these results appear to be particularly suitable for studying both stability and stabilization problems for uncertain systems independently of delays, as analysis and synthesis methods have been developed in recent years based upon small gain conditions.

The remainder of this paper is organized as follows. In Section 2, we define the notion of asymptotic stability independent of delays. Moreover, we introduce preliminary results for the structured singular value. Using these simple results, we develop in Section 3 necessary and sufficient conditions for asymptotic stability independent of delays. To streamline our presentation, we consider first systems with a single delay, whose development captures the essence of our general approach. Systems with multiple commensurate and non-commensurate delays are treated next. For systems with non-commensurate delays, the stability condition amounts to computing a structured singular value, while for systems with commensurate delays it suffices to compute a spectral radius, both on a pointwise basis in frequency. With this accomplished, we demonstrate in Section 4 how these conditions may be extended readily to study asymptotic stability independent of delays for uncertain systems. Finally, Section 5 contains a conclusion, wherein a brief discussion on stabilization problems is given.

2 Preliminaries

The linear time-invariant delay systems under consideration in this paper can be described by

$$\dot{x}(t) = Ax(t) + \sum_{k=0}^{q} B_k x(t - h_k), \qquad h_k \geq 0, \qquad (2.1)$$

where A, $B_k \in \mathbf{R}^{n\times n}$ are system matrices, and h_k are delay constants. Let $h_k = kh$, $k = 1, \cdots, q$ for some $h \geq 0$. Then, the delay factors h_k are said to be *commensurate*. Otherwise, the delays are said to be *non-commensurate* [14]. An interesting class of non-commensurate delays

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arise when the delays are each an independent parameter. The notion of asymptotic stability independent of delays [14] is defined as follows.

Definition 2.1 *The system (2.1) is said to be asymptotically stable independently of delays if*

$$\det\left(sI - A - \sum_{k=1}^{q} B_k e^{-h_k s}\right) \neq 0, \quad \forall s \in \overline{\mathbf{C}}_+, \quad \forall h_k \geq 0 \tag{2.2}$$

where $\mathbf{C}_+ := \{s : \, Re(s) > 0\}$ *denotes the open right half plane,* $\overline{\mathbf{C}}_+$ *denotes its closure, and* I *denotes the identity matrix of an appropriate dimension*[1].

Stated in words, the system (2.1) is asymptotically stable independently of delays if it is asymptotically stable with all nonnegative delays. Alternatively, this is equivalent to that its characteristic polynomial (2.2) has no roots in the closed right half plane.

We shall introduce below an important measure known as the *structured singular value* initially introduced in [6], [24]; this notion plays a crucial role in our subsequent development. Consider first, for a given matrix $M \in \mathbf{C}^{m \times m}$, a corresponding *block structure* [6] of q-tuple positive integers

$$\mathcal{K} := (k_1, \, \cdots, \, k_q), \qquad m = \sum_{i=1}^{q} k_i.$$

Further, consider the following family of diagonal matrices partitioned compatibly to $\mathcal{K}$:

$$\mathcal{X}_q(\gamma) := \left\{ \operatorname{diag}\left(\delta_1 I_{k_1} \, \cdots \, \delta_q I_{k_q}\right) : \, \delta_k \in \mathbf{C}, \, |\delta_k| \leq \gamma \right\}.$$

Definition 2.2 *The structured singular value of* M *with respect to* $\mathcal{X}_q(\gamma)$ *is defined as zero if* $\det(I - M\Delta) \neq 0 \; \forall \Delta \in \mathcal{X}_q(\infty)$ *and otherwise*

$$\mu_{\mathcal{X}_q}(M) := \left(\min_{\Delta \in \mathcal{X}_q(\infty)} \{\overline{\sigma}(\Delta) : \, \det(I - M\Delta) = 0\} \right)^{-1},$$

where $\overline{\sigma}(\Delta)$ *denotes the largest singular value of* Δ.

Note that in this definition we have tailored the set $\mathcal{X}_q(\infty)$ to one which contains only complex scalars. A more general framework can be found in [7].

Several relevant properties of the structured singular value will now be adapted below; a lucid discussion of these properties may be found in [6], [22], [2]. Define the sets

$$\mathcal{D} := \left\{ \operatorname{diag}\left(D_1 \, \cdots \, D_q\right) : \, D_i \in \mathbf{C}^{k_i \times k_i}, \, D_i = D_i^H \right\}$$

and

$$\mathcal{Q} := \left\{ \operatorname{diag}\left(\delta_1 I_{k_1} \, \cdots \, \delta_q I_{k_q}\right) : \, \delta_k \in \mathbf{C}, \, |\delta_k| = 1 \right\}.$$

For a given matrix M, let $\rho(M)$ denote its spectral radius. The following results are known in [22].

Fact 2.1

$$\mu_{\mathcal{X}_q}(M) = \max_{Q \in \mathcal{Q}} \rho(MQ). \tag{2.3}$$

Fact 2.2

$$\mu_{\mathcal{X}_q}(M) \leq \inf_{D \in \mathcal{D}} \overline{\sigma}\left(DMD^{-1}\right). \tag{2.4}$$

Furthermore, the upper bound is equal to $\mu_{\mathcal{X}_q}(M)$ *when* $q = 1$.

Additionally, the result below is taken from [2].

Fact 2.3 *Let* $M(s) \in \mathbf{C}^{m \times m}$ *be analytic in* $\overline{\mathbf{C}}_+$. *Then, both* $\rho(M(s))$ *and* $\mu_{\mathcal{X}_q}(M(s))$ *are continuous and subharmonic in* $\overline{\mathbf{C}}_+$.

The implication of this result is that if $M(s)$ is analytic in $\overline{\mathbf{C}}_+$, then the maximum of $\rho(M(s))$ and $\mu_{\mathcal{X}_q}(M(s))$ in $\overline{\mathbf{C}}_+$ will be achieved on the boundary of $\overline{\mathbf{C}}_+$, i.e., the imaginary axis. Furthermore, if $M(s)$ is not a constant matrix, then the maximum of $\rho(M(s))$ and $\mu_{\mathcal{X}_q}(M(s))$ can only be achieved on the imaginary axis.

Finally, the following explicit expression of the structured singular value can be derived as in [5] when M is a rank-one matrix.

Fact 2.4 *Let* $M = ba^H$ *with* $a, b \in \mathbf{C}^m$ *partitioned compatibly with* $\mathcal{X}_q(\gamma)$. *Then,*

$$\mu_{\mathcal{X}_q}(M) = \sum_{k=1}^{q} |a_k^H b_k|.$$

3 Stability Independent of Delays

In this section we develop necessary and sufficient conditions under which the system (2.1) will be asymptotically stable independently of delays. In doing so, we shall examine first the systems with a single delay, and later extend our investigation to systems with multiple delays.

3.1 Systems with a Single Delay

medskip To outline the key ideas in our derivation, it is instructive to consider first the systems with a single delay. In this section we attempt to do so by examining specifically the delay system

$$\dot{x}(t) = A \, x(t) + B \, x(t-h), \qquad h \geq 0. \tag{3.1}$$

First, the following necessary condition is known (see, e.g., [10], [13], [20]).

Lemma 3.1 *Suppose that the system (3.1) is asymptotically stable independently of delay. Then,* A *is asymptotically stable.*

Based upon Lemma 3.1, the following necessary and sufficient condition is obtained. This result may be considered as an extended small gain condition.

Theorem 3.1 *The system (3.1) is asymptotically stable independently of delay if and only if (i)* A *is asymptotically stable, (ii)*

$$\rho\left((j\omega I - A)^{-1} B\right) < 1, \qquad \forall \omega > 0. \tag{3.2}$$

and (iii) either 1) $\rho\left(A^{-1}B\right) < 1$ *or 2)* $\rho\left(A^{-1}B\right) = 1$ *and* $\det(A + B) \neq 0$.

Proof. To show the sufficiency, assume that A is asymptotically stable. Then, $(sI - A)^{-1}B$ is analytic on $\overline{\mathbf{C}}_+$, so is $(sI - A)^{-1}Be^{-hs}$ for any $h \geq 0$. Since $\rho\left((sI - A)^{-1}Be^{-hs}\right)$ is a subharmonic function on $\overline{\mathbf{C}}_+$, it follows that

$$\sup_{s \in \overline{\mathbf{C}}_+} \rho\left((sI - A)^{-1}Be^{-hs}\right) = \sup_{\omega \geq 0} \rho\left((j\omega I - A)^{-1}B\right).$$

Additionally, the maximum of $\rho\left((sI - A)^{-1}Be^{-hs}\right)$ cannot be achieved at any interior point of $\mathbf{C}_+$. As a result, the conditions (ii) and (iii-1) imply that

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[1] Occasionally, we shall use a subscript to indicate the dimension of the identity matrix I. This subscript will be suppressed, however, whenever the dimension is clear from the context.

$\rho\left((sI-A)^{-1}Be^{-hs}\right) < 1$ for all $s \in \overline{\mathbb{C}}_+$ and for all $h \geq 0$. This leads to that $\det\left(I-(sI-A)^{-1}Be^{-hs}\right) \neq 0$ for all $s \in \overline{\mathbb{C}}_+$ and for all $h \geq 0$. The latter condition, however, is equivalent to

$$\det\left(sI-A-Be^{-hs}\right) \neq 0, \qquad \forall s \in \overline{\mathbb{C}}_+,\ \forall h \geq 0. \quad (3.3)$$

Hence, the system (3.1) is asymptotically stable independently of delay. Alternatively, the conditions (ii) and (iii-2) imply that $\rho\left((sI-A)^{-1}Be^{-hs}\right) < 1$ for all $h \geq 0$ and for all $s \in \overline{\mathbb{C}}_+$, $s \neq 0$, which in turn leads to the condition (3.3) for all $h \geq 0$ and for all $s \in \overline{\mathbb{C}}_+$, $s \neq 0$. At $s = 0$, the condition (3.3) also holds since $\det(A+B) \neq 0$. Hence again, the system (3.1) is asymptotically stable independently of delay. This completes the proof for sufficiency. To establish the necessity, it suffices, in light of Lemma 3.1, to show that (ii) and (iii) are necessary. Toward this end, assume that $\rho\left((j\omega I-A)^{-1}B\right) \geq 1$ for some $\omega \geq 0$. Consider first the case $\rho\left((j\omega I-A)^{-1}B\right) = 1$. If $\omega = 0$, then $\rho\left(A^{-1}B\right) = 1$, and the condition (iii-2) is clearly necessary. Otherwise, let $e^{j\alpha}$ be an eigenvalue of $(j\omega I-A)^{-1}B$. Select $h \geq 0$ such that $h = \alpha/\omega$ if $\alpha \geq 0$ and $h = (2\pi+\alpha)/\omega$ if $\alpha < 0$. It follows that $\det\left(I-(j\omega I-A)^{-1}Be^{-jh\omega}\right) = 0$, or equivalently, $\det\left(j\omega I-A-Be^{-jh\omega}\right) = 0$. This contradicts to the fact that (3.1) is asymptotically stable. Next, consider the case $\rho\left((j\omega I-A)^{-1}B\right) > 1$. Since $\rho\left((j\omega I-A)^{-1}B\right) = 0$ when $\omega \to \infty$, and since $\rho\left((j\omega I-A)^{-1}B\right)$ is a continuous function of ω, it follows that $\rho\left((j\omega_0 I-A)^{-1}B\right) = 1$ for some $\omega_0 \geq 0$. This, however, has been discussed above. The proof is now completed. ■

The stability condition given in Theorem 3.1 is clearly an improvement over that reported in [10], which essentially is similar to and has the same level of difficulty as the sufficient conditions of [13], [14], [4], stated in terms of two-variable polynomials. This result is useful in that it provides a test that can be readily checked. Indeed, to implement the test one needs to compute essentially only the maximum of $\rho\left((j\omega I-A)^{-1}B\right)$.

As an immediate consequence, the above necessary and sufficient condition yields the following sufficient condition: The system (3.1) is asymptotically stable independently of delay if (i) A is asymptotically stable, and (ii)

$$\rho\left((j\omega I-A)^{-1}B\right) < 1, \qquad \forall \omega \geq 0. \quad (3.4)$$

This condition differs from that in Theorem 3.1 in that the condition $\rho\left(A^{-1}B\right) < 1$ is included in (3.4). Under the condition that A is asymptotically stable, one can show readily that (3.4) is equivalent to the two-variable criterion. Indeed, in the two-variable criterion, one needs to check the condition [13], [14]

$$\det\left(sI-A-Bz\right) \neq 0, \qquad \forall s \in \overline{\mathbb{C}}_+,\ \forall |z| = 1, \quad (3.5)$$

which in turn is equivalent to that

$$\det\left(I-(sI-A)^{-1}Bz\right) \neq 0$$

for all $s \in \overline{\mathbb{C}}_+$ and for all $|z| \leq 1$. However, from the definition of the structured singular value, we conclude that the latter and hence (3.5) is equivalent to the condition $\mu_{\mathcal{X}_1}\left((sI-A)^{-1}B\right) < 1$ for all $s \in \overline{\mathbb{C}}_+$. In light of Fact 2.1 and Fact 2.3, we further conclude that (3.5) is equivalent to (3.4). As such, we see that both the condition (3.4) and the two-variable criterion can be interpreted as a small gain condition using the structured singular value. It turns out that this interpretation has an important implication toward why both these conditions lose necessity. To see this, we observe that the necessity of a small gain condition such as $\mu_{\mathcal{X}_q}(M) < 1$ hinges on the existence of an "uncertainty" $\Delta \in \mathcal{X}_q(1)$ with which the matrix $I - M\Delta$ loses its rank, and this underlines precisely our above derivation. Indeed, in establishing the necessity of above necessary and sufficient condition, a key has been to construct a delay parameter h, or a $\delta = e^{-hs}I$ such that the matrix $I-(sI-A)^{-1}Be^{-hs}$ loses rank, and this construction is allowed for $s \neq 0$ because $h \geq 0$ can be selected arbitrarily due to the nature of asymptotic stability independent of delays. However, for $s = 0$, such a construction of Δ is not possible despite that h can be selected arbitrarily. This is precisely where the conditions (3.4) and (3.5) lose the necessity. Note that by imposing an additional condition $\det(A+Bz) \neq 0$ for all $|z| = 1$ in Definition 2.1, as suggested in [15], then, both the conditions (3.4) and (3.5) become necessary. This extra condition is clearly equivalent to $\rho\left(A^{-1}B\right) < 1$, and it restricts the class of systems which may be considered.

Finally, a more conservative sufficient condition may be obtained by weakening (3.4) to

$$\sup_{\omega \geq 0} \overline{\sigma}\left((j\omega I-A)^{-1}B\right) < 1.$$

This result was given in [26], and it requires computing the H_∞ norm of the transfer function $(sI-A)^{-1}B$. Though it may be arbitrarily weaker than (3.4), this result, as noted in [26], can nonetheless be verified more easily. Note also that if A and B are scalars, $A = a < 0$, $B = b$, then,

$$\rho\left((j\omega I-A)^{-1}B\right) = \left|(j\omega I-A)^{-1}B\right| = \frac{|b|}{\sqrt{\omega^2+a^2}}.$$

It follows that the system (3.1) is asymptotically stable independently of delay if and only if either $-a > |b|$ or $a = b$. The former case has been shown in [13]. However, the latter case was excluded therein due to the fact that the two-variable criterion was used. An example of this case was given in [15] to show that the two-variable criterion in general is not a necessary condition.

3.2 Systems with Multiple Delays

We shall now extend the preceding results to systems with multiple delays. Our derivation below follows essentially the same approach and constitutes a natural extension of that in the previous section. We shall consider first the systems with *non-commensurate* delays. Note that with both commensurate and non-commensurate delays, the system (2.1) will be asymptotically stable independently of delays only when A is asymptotically stable.

Theorem 3.2 *Suppose that in (2.1) $h_k \geq 0$ are independent delays for all $k = 1, \cdots, q$. Then, the system (2.1) is asymptotically stable independently of delays if and only if (i) A is asymptotically stable, (ii)*

$$\mu_{\mathcal{X}_q}\left(M(j\omega)\right) < 1, \qquad \forall \omega > 0, \quad (3.6)$$

and (iii) either 1) $\mu_{\mathcal{X}_q}(M(0)) < 1$ or 2) $\mu_{\mathcal{X}_q}(M(0)) = 1$ and $\det\left(I - M(0)\right) \neq 0$, *where $M(s) \in \mathbb{C}^{nq \times nq}$ is defined as*

$$M(s) := \begin{bmatrix} I & \cdots & I \end{bmatrix}^T (sI-A)^{-1} \begin{bmatrix} B_1 & \cdots & B_q \end{bmatrix}.$$

Proof. Suppose that A is asymptotically stable. Then, the system (2.1) is asymptotically stable independently of delay if and only if $\forall s \in \overline{\mathbb{C}}_+$, $\forall h_k \geq 0$,

$$\det\left(I-(sI-A)^{-1}\sum_{k=1}^{q} B_k e^{-h_k s}\right) \neq 0. \quad (3.7)$$

Let $\Delta(s) := \operatorname{diag}\left(e^{-h_1 s}I \ \cdots \ e^{-h_q s}I\right)$. It follows that (3.7) is satisfied if and only if $\det(I - M(s)\Delta(s)) \neq 0$ for all $s \in \overline{\mathbf{C}}_+$ and for all $h_k \geq 0$. Note that as in the proof of Theorem 2.1,

$$\sup_{s\in\overline{\mathbf{C}}_+} \rho(M(s)\Delta(s)) = \sup_{\omega\geq 0} \rho(M(j\omega)\Delta(j\omega)).$$

Since h_k are arbitrary independent real constants, we may replace $\Delta(j\omega)$ by an arbitrary constant matrix $Q \in \mathcal{Q}$ for all $\omega > 0$. As a result,

$$\begin{aligned} \sup_{h_k\geq 0}\sup_{\omega>0} \rho(M(j\omega)\Delta(j\omega)) &= \sup_{\omega>0}\sup_{Q\in\mathcal{Q}} \rho(M(j\omega)Q) \\ &= \sup_{\omega>0} \mu_{\mathcal{X}_q}(M(j\omega)). \end{aligned}$$

Hence, if the conditions (ii) and (iii-1) are satisfied, the condition (3.7) will hold. Alternatively, the conditions (ii) and (iii-2) imply that for all $h_k \geq 0$, $\rho(M(s)\Delta(s)) \leq 1$ for all $s \in \overline{\mathbf{C}}_+$, and in particular, $\rho(M(s)\Delta(s)) < 1$ for all $s \in \overline{\mathbf{C}}_+$ such that $s \neq 0$. This suggests that (3.7) holds for all $s \in \overline{\mathbf{C}}_+$ such that $s \neq 0$. Moreover, since $\det(I - M(0)) \neq 0$, it follows that (3.7) holds at $s = 0$ as well. Consequently, the conditions (i)-(iii) imply that the system (2.1) is asymptotically stable independently of delay. The proof for the sufficiency part is completed. To show the necessity, assume first that $\mu_{\mathcal{X}_q}(M(j\omega)) = 1$ for some $\omega > 0$. Then, there exists $Q = \operatorname{diag}\left(e^{j\alpha_1}I_n \ \cdots \ e^{j\alpha_q}I_n\right)$ such that $\det(I - M(j\omega)Q) = 0$. A set of delay constants may then be constructed as $h_k = \alpha_k/\omega$ if $\alpha_k \geq 0$ and $h_k = (2\pi + \alpha_k)/\omega$ if $\alpha_k < 0$, so that the condition (3.7) is violated at $s = j\omega$. If, however, $\mu_{\mathcal{X}_q}(M(j\omega)) > 1$ for some $\omega > 0$, then by the continuity of $\mu_{\mathcal{X}_q}(M(j\omega))$, and also by the fact that $\lim_{\omega\to\infty} \mu_{\mathcal{X}_q}(M(j\omega)) = 0$, there exists some $\omega_0 \in (\omega, \infty)$ such that $\mu_{\mathcal{X}_q}(M(j\omega_0)) = 1$. Again, in this case, the condition (3.7) will be violated at $s = j\omega_0$. Hence, in order for (3.7) to hold, it is necessary that (ii) holds. Finally, we claim that it is also necessary that $\mu_{\mathcal{X}_q}(M(0)) \leq 1$, hence establishing the necessity of (iii). Indeed, if $\mu_{\mathcal{X}_q}(M(0)) > 1$, then by continuity there exists $\omega > 0$ such that $\mu_{\mathcal{X}_q}(M(j\omega)) = 1$, and as shown above, the condition (3.7) will be violated at $s = j\omega$. The proof is now completed. ■

Again, in the case of non-commensurate delay systems, Theorem 3.2 provides an extended small gain condition for the asymptotic stability of (2.1) independent of delays. This condition can be checked by computing a structured singular value. Note, however, that an exact method in computing the structured singular value is not currently available [22]. Nevertheless, a tight estimate on the structured singular value may still be obtained by computing the upper bound in (2.4), which, as shown in [22], may be computed efficiently. Also, following from Fact 2.4, $\mu_{\mathcal{X}_q}(M(j\omega))$ can be determined explicitly in the special case where the matrix $[\ B_1 \ \cdots \ B_q\ ]$ possesses a rank of one. Indeed, write $[\ B_1 \ \cdots \ B_q\ ] = uv^T$, where $u \in \mathbf{R}^n$ and $v = [\ v_1^T \ \cdots \ v_q^T\ ]^T \in \mathbf{R}^{nq}$. Then,

$$\mu_{\mathcal{X}_q}(M(j\omega)) = \sum_{k=1}^{q} \left|v_k^T(j\omega I - A)^{-1}u\right|.$$

This expression can be further simplified if A and B_k are all scalars. Let $A = a < 0$ and $B_k = b_k$. Then, $u = 1$ and $v_k = b_k$, and one obtains

$$\mu_{\mathcal{X}_q}(M(j\omega)) = \frac{\sum_{k=1}^{q}|b_k|}{\sqrt{\omega^2 + a^2}}.$$

Consequently, the system (2.1) will be asymptotically stable independently of delays if and only if $-a > \sum_{k=1}^{q}|b_k|$, or $-a = \sum_{k=1}^{q}|b_k|$ but $a + \sum_{k=1}^{q} b_k \neq 0$.

Next, we give a necessary and sufficient condition for the asymptotic stability independent of delays for the system (2.1) with commensurate delays. The key step in our derivation is to convert the stability problem into one with respect to a single delay.

Theorem 3.3 *Suppose that in (2.1) $h_k = kh$ for some $h \geq 0$. Then, the system (2.1) is asymptotically stable independently of delays if and only if (i) A is asymptotically stable, (ii)*

$$\rho(M_q(j\omega)) < 1, \qquad \forall\omega > 0, \tag{3.8}$$

and (iii) either 1) $\rho(M_q(0)) < 1$ or 2) $\rho(M_q(0)) = 1$ and $\det(I - M_q(0)) \neq 0$, *where $M_q(s)$ is defined as follows:*

$$M_1(s) := [\ I \ \cdots \ I\ ]^T (sI - A)^{-1} [\ B_1 \ \cdots \ B_q\ ],$$

$$m_1 := n, \qquad m_{k+1} := m_k + (q - k + 1)n,$$

$$M_{k+1}(s) := \begin{bmatrix} 0 & I_{(q-k)n} \\ M_k P_k & 0 \end{bmatrix}, \ P_k := \begin{bmatrix} 0 & I_{m_k} \\ I_{(q-k)n} & 0 \end{bmatrix}$$

Proof. see the full version of this paper. ■

4 Robust Stability Independent of Delays

Thus far we have derived necessary and sufficient conditions regarding asymptotic stability independent of delays for systems with both commensurate and non-commensurate delays. An important feature of our results is that they are extended small gain conditions stated specifically in terms of certain structured singular values. These conditions are particularly suitable for studying robust stability problems, among other problems. In this section, we demonstrate how the preceding results may be extended to *uncertain* delay systems. For purpose of illustration, we shall consider only systems with a single delay; however, similar results can be derived analogously for systems with multiple delays. Also, for ease of presentation, we shall assume that the uncertain system is of the form

$$\dot{x}(t) = (A + \Delta_A)x(t) + (B + \Delta_B)x(t-h), \quad h \geq 0, \tag{4.1}$$

where Δ_A and Δ_B represent the uncertainties in the system and are bounded as $\overline{\sigma}(\Delta_A) \leq \gamma_A$ and $\overline{\sigma}(\Delta_B) \leq \gamma_B$, respectively. Though this specific class of uncertainties may appear restrictive, it will become evident that the technique used here can be extended readily to systems with other types of more structured uncertainties.

Our purpose is to derive a necessary and sufficient condition for *robust* asymptotic stability independent of delay, by which we mean that the system (4.1) is asymptotically stable independently of delay for all Δ_A and Δ_B such that $\overline{\sigma}(\Delta_A) \leq \gamma_A$ and $\overline{\sigma}(\Delta_B) \leq \gamma_B$. For this purpose, a more general framework of the structured singular value is needed. Define, in addition to the set $\mathcal{X}_q(\gamma)$,

$$\mathcal{Y}_k(\gamma) := \left\{\operatorname{diag}(\Delta_1 \ \cdots \ \Delta_k) : \ \Delta_i \in \mathbf{R}^{n\times n}, \ \overline{\sigma}(\Delta_i) \leq \gamma\right\},$$

and

$$\mathcal{Z}_k(\gamma) := \left\{\operatorname{diag}(\Delta_1 \ \Delta_2) : \ \Delta_1 \in \mathcal{X}_1(\gamma), \ \Delta_2 \in \mathcal{Y}_k(\gamma)\right\}.$$

Also, define analogously the structured singular values $\mu_{\mathcal{Y}_k}(\cdot)$ and $\mu_{\mathcal{Z}_k}(\cdot)$, with respect to $\mathcal{Y}_k(\gamma)$ and $\mathcal{Z}_k(\gamma)$, respectively. Note that in order for the system (4.1) to be robustly asymptotically stable independently of delay, it is necessary that both A and $A+B$ are asymptotically stable.

Theorem 4.1 *Suppose that A and $A+B$ are asymptotically stable. Then, the system (4.1) is robustly asymptotically stable independently of delay if and only if (i)*

$$\mu_{\mathcal{Z}_2}(M(j\omega)) < 1, \qquad \forall \omega \geq 0, \tag{4.2}$$

or (ii) $\mu_{\mathcal{Z}_2}(M(j\omega)) < 1$ for all $\omega > 0$, $\mu_{\mathcal{Z}_2}(M(0)) = 1$, and $\mu_{\mathcal{Y}_2}(N) < 1$, where

$$M(s) := \begin{bmatrix} (sI-A)^{-1}B & (sI-A)^{-1} & (sI-A)^{-1} \\ -\gamma_B I & 0 & 0 \\ \gamma_A(sI-A)^{-1}B & \gamma_A(sI-A)^{-1} & \gamma_A(sI-A)^{-1} \end{bmatrix}$$

$$N := \begin{bmatrix} I & I \end{bmatrix}^T (A+B)^{-1} \begin{bmatrix} \gamma_A I & \gamma_B I \end{bmatrix}.$$

Proof. See the full version of this paper. ■

Hence again, the robust asymptotic stability of (4.1) independent of delay can be, in principle, determined by computing a structured singular value as well. It should be noted that the exact computation of such a structured singular value remains to be an open problem. Nevertheless, by computing an upper bound such as that in (2.4), one may obtain a potentially non-conservative sufficient condition.

5 Conclusion

The approach as well as the results presented in this paper should constitute a useful extension to the previously known results. A notable improvement achieved by these results lies in the fact that they can be readily implemented; indeed, stated in terms of structured singular values, these conditions may be checked efficiently by making use of the well-documented softwares developed for robustness analysis in general and for structured singular value analysis in particular (see, e.g., [1]). This is especially the case for systems with commensurate delays, for which the presented test amounts to computing the spectral radius of a frequency dependent matrix. It is worth noting that these necessary and sufficient conditions are even considerably simpler than the available sufficient conditions in the spirit of [14] and [4], which have the same level of difficulty as that in [10]. Additionally, as small gain conditions, these results appear to be particularly suitable for the purpose of studying robust stability properties, and further are suited for studying stabilization problems for delay systems. In this vein, one notes that for the class of *stabilizable* systems independently of delay [14], [16], [8], the standard H_∞ synthesis and quadratic stabilization methods may be employed to synthesize a stabilizing compensator. Application of these methods to stabilization of delay systems have been reported in e.g., [26], [23]. Also, at the expense of an increased level of computation effort, the structured singular value synthesis [1] may be used to synthesize an "optimal" compensator such that the closed loop system is stable independently of delays.

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