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The Critical Perturbation Radius Weights in $\mathcal{H}_\infty$ Synthesis for Interval Plants

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Abstract

In this paper the $\mathcal{H}_\infty$ synthesis of robust controllers for interval plants is considered. Generally, the uncertainties which the $\mathcal{H}_\infty$ method deals with are modeled as non-parametric (norm-bounded) uncertainties, and a weighting function is used to specify the frequency-selective effects of the uncertainties. The first attempt to deal with the synthesis problem for a SISO interval plant using $\mathcal{H}_\infty$ methods, used the Maximum Perturbation Radius (MPR) as the weight. Unfortunately, the MPR which only yields a sufficient stability condition, not only introduces conservatism but also becomes computationally inefficient for general interval perturbations. The more recently introduced Critical Perturbation Radius (CPR) yields necessary and sufficient conditions for robust stability and is thus an ideal candidate for a systematic methodology in defining non-conservative $\mathcal{H}_\infty$ uncertainty weights. This paper illustrates the application of the CPR weights and shows that significant improvements can be obtained by using the Critical Pertrubation Radius rather than the Maximum Perturbation Radius.

1. Introduction

In recent years there has been much progress in the analysis of the stability properties of systems in the presence of parametric uncertainties. (See for example [16],[17],[3] and [2].) Of particular interest in the context of this work are the results which provide stability results using the frequency response of the nominal system together with uncertainty templates in the frequency domain. Unfortunately, for general classes of interval plants, the uncertainty templates in the frequency domain are irregular in shape and this has resulted in conservative stability results. For example, it is common to replace the irregularly shaped uncertainty templates by an over bounding disk whose radius (its Maximum Pertrubation Radius (MPR)) is precisely the largest deviation, at each frequency, of the uncertainty template (in any direction) from the nominal frequency response point.

The first attempt to use $\mathcal{H}_\infty$ synthesis methods for a SISO interval plant was made in [1]. There, a Maximum Pertrubation Radius is selected as the uncertainty weight. However, the results in [1] not only introduce conservatism but apply to one restricted uncertainty - the case of linear interval perturbations. Moreover, for general interval uncertainties, the computation of the MPR turns out to be tedious and more importantly, the MPR only gives sufficient conditions for robust stability.

In this paper we use the recently proposed Critical Direction Method (CDM), which yields necessary and sufficient stability conditions. It is applicable to a large class of important interval plants. Instead of using the MPR defined earlier, the CDM approach uses the Critical Perturbation Radius (CPR) which is a measure of the deviation from the nominal frequency response of the uncertainty template along the "critical line" joining the nominal frequency response for the $(-1 + j0)$ point. Thus, the CPR exploits the directionality properties of the uncertainty template (which is surrounded by the MPR) to yield less conservative stability conditions. In the paper we use the CPR rather than the MPR for defining appropriate weights for an $\mathcal{H}_\infty$ controller synthesis for interval plants.

The rest of the paper is organized as follows. Section 2 presents some necessary background and preliminary results on the CPR and MPR. Section 3 gives the formulation of the CPR weighting for the design problem. In Section 4 we design $\mathcal{H}_\infty$ controllers using constant and frequency dependent CPR and MPR weights. Our results show that the CPR not only yields better robustness margins, but also gives better rejection of the effects of uncertainties as measured by the time response. Our conclusions and some comments on future work are given in Section 5.

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2. Preliminaries

In this section preliminary results and notation from [1] and [2], are reviewed. Consider the linear uncertain system

$$g(s) = g_0(s) + \delta(s) \quad (1)$$

where $g_0(s)$ is a known nominal system and $\delta(s)$ is a perturbation that represents the modeling error. The perturbation $\delta(s)$ belongs to an uncertainty-description set $\mathbf{d}$, which describes the allowable set of interval uncertainties.

2.1. Maximum Perturbation Radius Weight

The MPR of the value set is defined in [1], and illustrated in Figure 1. The largest perturbation radius for each frequency $MPR(\omega)$ is defined as

$$MPR(\omega) = \delta_{\max}(j\omega) = \max_{\mathbf{d}} |g(j\omega) - g_0(j\omega)| \quad (2)$$

The largest $MPR(\omega)$ over all frequencies will be denoted by the constant,

$$MPR \triangleq \max_{\omega} |\delta_{\max}(j\omega)| = \|\delta_{\max}(j\omega)\|_\infty. \quad (3)$$

For the linear perturbation case, at eachfrequency the MPR in the family of transfer functions $g(s)$, will correspond to a point on one of the extreme segments, which is defined by the Kharitonov segments [1]. Searching for the MPR for the linear perturbation case is very simple for there are only at most sixteen extreme segments to be considered. For other types of perturbations the extreme segments are not as well defined and the search for the MPR becomes quite tedious. Moreover, since the MPR weight only provides sufficient conditions for robust stability conditions for robustness is desired. The CPR method

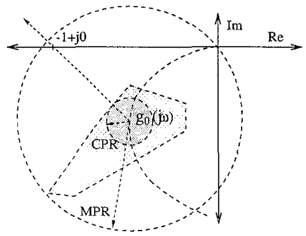

Figure 1: Conservativeness of Maximum Radius

yields an efficient and non-conservative robust stability result which makes it an ideal candidate for generating a more appropriate $\mathcal{H}_\infty$ weight.

2.2. Critical Direction and Perturbation Radius

Latchman and Crisalle [2] propose a frequency-domain robustness analysis technique, the critical direction theory, that makes use of the following objects:

1. The nominal frequency-response $g_0(j\omega)$.
2. The *critical line,* defined on the directed line which originates at the nominal point $g_0(j\omega)$ and passes through the critical point $-1 + j0$.
3. The *Critical Direction*

$$d(j\omega) = -\frac{1 + g_0(j\omega)}{|1 + g_0(j\omega)|}. \tag{4}$$

   The unit vector that defines the direction of critical line.

4. The *Uncertainty Template* (or value set)

$$\mathcal{T}(\omega) = \{g(j\omega)|g(j\omega) = g_0(j\omega) + \delta(j\omega), \delta(s) \in \mathbf{d}\}. \tag{5}$$

5. The *Critical Uncertainty Template*

$$\mathcal{T}_c(\omega) = \{g(j\omega) \in \mathcal{T}(\omega)|g(j\omega) = g_0(j\omega) + \alpha d(j\omega)\},$$

$$\textit{for some } \alpha \in \mathcal{R}_+. \tag{6}$$

6. The *Critical Perturbation Radius* (CPR) ttfamily

$$\rho_c(\omega) = \max_{\alpha \in \mathcal{R}_+} \{\alpha | z = g_0(j\omega) + \alpha d(j\omega) \in \mathcal{T}_c(\omega)\}. \tag{7}$$

7. The *Critical Locus*

$$h_c(j\omega) = g_0(j\omega) + \rho_c(\omega) d(j\omega). \tag{8}$$

At every frequency, $\rho_c(j\omega)$ represents the element of $T_c(\omega)$ that is closest to the point $-1+j0$ along the critical direction.

Note that at every frequency ω, the critical direction $d(j\omega)$ may be interpreted as a unit vector with origin at $g_0(j\omega)$ and pointing towards the point $-1+j0$. Also note that δ_c represents the set of perturbations with frequency response lying along a straight-line segment that joins the points $g_0(j\omega)$ and $-1 + j0$. The critical radius is illustrated in Figure 2, which shows that $\rho_c(\omega)$ is determined by the intersection of the boundary of the template with the straight-line segment that joins $g_0(j\omega)$ and $-1 + j0$. The necessary and sufficient stability condition of the CPR is given in the next theorem. Figure 1 illustrates the CPR and MPR and Figure 2 shows the critical direction for typical uncertainty templates.

Theorem 1 *[2] Consider the uncertain system (1) and suppose that the nominal system $g_0(s)$ is stable under unity feedback, and that $g(s)$ and $g_0(s)$ have the same number of open-loop unstable poles. Then the uncertain system is stable under unity feedback if and only if*

$$\frac{\rho_c(\omega)}{|1 + g_0(j\omega)|} < 1, \forall \omega. \tag{9}$$

The calculation of $\rho_c(\omega)$ is based on the description of the boundary of the value set [2]. The CPR's dependence upon the value set makes the results in this paper applicable to general perturbation intervals such as multilinear, multiaffine and nonlinear perturbation ([3],[4],[5]). For simplicity, the discussion will be focused on affine interval descriptions of the type

$$g(s,q) = \frac{N(s,q)}{D(s,q)}, \tag{10}$$

where the coefficients of $N(s,q)$ and $D(s,q)$ are affine functions of $q = (q_1, q_2, ..., q_n) \in Q$, and Q is a hypercube. Also, let ∂Q denote the edges of Q, and $\partial g(j\omega, Q)$ denote the boundary of the value set in the Nyquist plane. Then the Critical Perturbation Radius (CPR) ρ_ω can be found by searching the edges of ∂Q which intersect the critical direction.

The collection of $\rho_c(\omega)$ over all frequencies forms a Critical Perturbation Locus. A rational transfer-function approximation $\rho_c(s)$ can then be found using a numerical inversion technique, such as the method of [7]. This $\rho_c(s)$ then can be used to define the uncertainty weight needed for the $\mathcal{H}_\infty$ synthesis method.

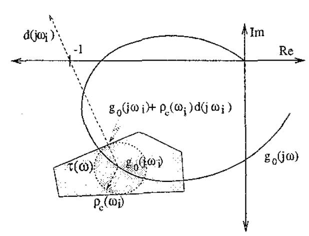

Figure 2: Critical Direction and Critical Radius

3. Critical Perturbation Radius Weight

Consider the uncertain system with unity negative feedback as shown in Figure 3. This classical feedback diagram can be converted to the now standard Linear Fractional Transformation (LFT) representation shown in Figure 4 and 5. The objective of $\mathcal{H}_\infty$ control is to minimize the effect of the uncertainty $\delta(s)$, on the output, which is accomplished by minimizing the ratio $\frac{\|z\|_2}{\|w\|_2}$, over all possible inputs, w. It turns out that

$$\sup_{w \epsilon L_2} \frac{||z||_2}{||w||_2} \triangleq ||T_{zw}(s)||_\infty = \sup_w |T_{zw}(\omega)|, \quad (11)$$

where L_2 is the set of square integrable functions and T_{zw} is the transfer function from w to z.

Thus, the objective of the $\mathcal{H}_\infty$ controller in this case is to minimize T_{zw}, where $z = T_{zw}(s)w$. From Figure 5 we have,

$$z = \delta(s)u = \delta(s)C(s)[w - g_0 u], \quad (12)$$

$$z = \delta(s)C(s)w - g_0(s)C(s)\delta(s)u, \quad (13)$$

$$z = \left[\frac{C(s)\delta(s)}{1 + g_0 C(s)}\right] w, \quad (14)$$

or

$$T_{zw} = \frac{C(s)\delta(s)}{1 + g_0 C(s)}. \quad (15)$$

Notice that without the controller, $(C(s) = 1)$

$$||T_{zw}(s)||_\infty = \left\| \frac{\delta(s)}{1 + g_0} \right\|_\infty, \quad (16)$$

which is identical in form to the expression appearing in Theorem 1,

$$\frac{\rho_c(\omega)}{|1 + g_0(j\omega)|} < 1, \forall \omega \quad (17)$$

or

$$\left\| \frac{\rho_c(\omega)}{1 + g_0(j\omega)} \right\|_\infty < 1, \quad (18)$$

where, $\rho_c(s)$ is the transfer function approximation of the Critical Perturbation Radius (CPR).

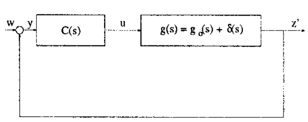

Figure 3: Uncertain System with Unity Negative Gain

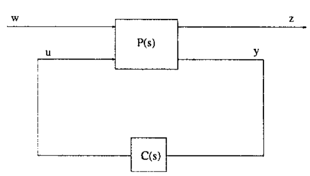

Figure 4: General Linear Fractional Transformation (LFT) Representation

Clearly, ρ_c is precisely the weighting function which captures the necessary and sufficient stability condition and thus can be used for non-conservative design.

For a given ρ_c obtained from a CPR analysis of the nominal system with $C(s) = 1$, we can design an $\mathcal{H}_\infty$, $C(s)$. In the next section we discuss the design procedure and give a comparison between the MPR and CPR based weighting strategies. Naturally, the controller, $C(s)$, will modify the orientation and size of the uncertainty template. This observation suggests that an iterative design procedure should be used to obtain the best results. This process would be similar to the now well known D-K iteration associated with μ - synthesis for structured uncertainties, and is the subject of ongoing research.

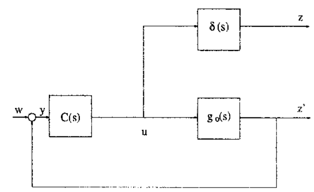

Figure 5: LFT Model for Interval Plant Feedback System

4. $\mathcal{H}_\infty$ Synthesis For Interval Systems

Consider the interval plant feedback system shown in Figure 3. This system can be modeled as a nominal plant with weighted perturbations, as shown in Figure 4, where $g(s,q)$ is an interval plant with parametric uncertainties described by q (Eq. 10). Figure 4 is the general setting for the $\mathcal{H}_\infty$ synthesis method. Now having found a suitable weight to effectively represent the perturbation of interval plants using the CPR method, the $\mathcal{H}_\infty$ synthesis problem for an interval plant can be solved.

In what follows we assume that the system matrix $P(s)$ in Figure 4 has the state-space representation:

$$\dot{x} = Ax + B_1 w + B_2 u \quad (19)$$

$$z = C_1 x + D_{12} u \quad (20)$$

$$y = C_2 x + D21w \quad (21)$$

4.1. $\mathcal{H}_\infty$ Solution By Two Ricatti Equation

Although the solution of $\mathcal{H}_\infty$ problem has numerous methods, we chose the simple 2-Ricatti equation for the purpose of comparison with the results found in [1]. Finding the $\mathcal{H}_\infty$ solution of 2-Ricatti equation begins with solving two Hamiltonian matrices of the form

$$H = \begin{bmatrix} A & -R \\ -Q & -A^T \end{bmatrix}, \quad (22)$$

with Q and R symmetric, but not necessarily positive definite. The associated Riccati equation is,

$$A^T P + PA - PRP + Q = 0. \quad (23)$$

The $\mathcal{H}_\infty$ solution requires solving 2-Riccati equations generated from two Hamiltonian matrices, which respectively represent the controllability and observability:

$$H_c = \begin{bmatrix} A & \gamma^2 B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix} \quad (24)$$

and

$$H_f = \begin{bmatrix} A^T & \gamma^2 C_1^T C_1 - C_2^T C_2 \\ -B_1 B_1^T & -A \end{bmatrix}. \quad (25)$$

For solving the Riccati equations associated with these two Hamiltonian matrices, there are several assumptions that are made about the Hamiltonian matrices [15][10]. Basically the solution P_c and P_f of Riccati equation associated with H_c and H_f are required to be positive semi-definite and $\max\{\lambda(P_cP_f)\} < \gamma^{-2}$. Then there exists a controller

$$C(s) \equiv \begin{bmatrix} A_c & B_c \\ F_c & 0 \end{bmatrix},$$

where

$$\begin{cases} A_c & = A + (\gamma^2 B_1 B_1^T - B_2 B_2^T)P_c \\ & \quad -(I - \gamma^2 P_c P_f)^{-1} P_f C_2^T C_2 \\ B_c & = (I - \gamma^2 P_c P_f)^{-1} P_f C_2^T \\ F_c & = -B_2^T P_c \end{cases} \quad (26)$$

The objective criterion of an $\mathcal{H}_\infty$ design is to limit the uncertainty effects on the system, such that there exists a necessary and sufficient condition for robust stability

$$\|T_{wz}(s)\|_\infty < \gamma.$$

The vector z represents the effects from w, and it should be kept small. The design objective is to keep the norm of the transmission, T_{zw} small. More precisely, the smaller γ is the better the design. In the following example four selected weights, as shown in Figure 6, are used to illustrate the effects of the constant and frequency dependent MPR and CPR weights.

Implementation Procedure Given an interval plant $g(s,q)$ where q is the interval in parametric space , the implementation procedure is outlined as follows:

1. Calculate the $\rho_c(\omega)$ for each frequency, then find a transfer function $h_c(j\omega) \cong \rho_c(\omega)$.
2. Derive the weighting function W.
3. Implement plant augmentation

$$\hat{P}(s) = \begin{bmatrix} A & \vdots & B_1 & B_2 \\ \cdots & \cdots & \cdots & \cdots \\ C_1 & \vdots & 0 & D_{12} \\ C_2 & \vdots & D_{21} & 0 \end{bmatrix}. \quad (27)$$

4. Solve the 2-Ricatti equation and obtain the controller from equation (26).

4.2. Example

Given an interval plant [1]

$$P(s,q) = \frac{(5s + q_1)}{s^2 + q_2 s + q_3},$$

with intervals

$q_1 \in [4-\epsilon, 4+\epsilon]$, $q_2 \in [2-\epsilon, 2+\epsilon]$, $q_1 \in [-15-\epsilon, -15+\epsilon]$

The state-space representation and uncertainty input-output matrices are:

$$A = \begin{bmatrix} -2 & 15 \\ 1 & 0 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$\begin{aligned} C_1 &= [0,0]; \quad C_2 = [-5,-4]; \\ D_{11} &= 0; \quad D_{12} = r; \quad D_{21} = 1; \quad D_{22} = 0; \end{aligned}$$

The largest value of the perturbation, ϵ (namely the stability margin), may be computed using the sixteen plant Kharitonov model as in [1], to get ϵ=2.83. For this value of ϵ , the constant $W_{MPR} \triangleq max_w MPR(w) =$ 0.395.

In addition, for ϵ = 2.83 we also calculate the frequency dependent $MPR(w)$ for a range of frequencies and obtain a transfer function

$$W_{MPR}(s) = \frac{0.14(s + 0.06)}{(s + 1.3)(s + 2.4)}; \quad (28)$$

as the dynamic MPR weighting function. Using the CDM and the bisection algorithm described in [18] we obtain the Critical Pertrubation Radius and the CPR weighting function

$$W_{CPR}(s) = \frac{0.0172s^2 + 0.2135s + 1.0163}{s^2 + 2.3805s + 5.7462}. \quad (29)$$

To facilitate comparison with the constant MPR signal weight W_{MPR}, we also calculate the constant CPR weight

$$W_{CPR} = \max_w |\rho_c(w)|. \quad (30)$$

Figure 7 shows $\rho_c(w)$ and its functional approximation $W_{CPR}(s)$. $\mathcal{H}_\infty$ controllers were obtained using $D_{12} = W_{MPR}$ and W_{CPR} as well as by using $W_{MPR}(s)$ and $W_{CPR}(s)$ in the plant augmentation. Table 1 shows the resulting $\|T_{zw}\|_\infty$ in each case and Figure 8 shows the corresponding frequency response of $T_{zw}(s)$ (To facilitate comparison, in all cases we take $\gamma = 1$). At the same peak level, the dynamic weight design shows a better capability to cope with the uncertainty. We find that the design with the overbounded constant weight does not result in a better uncertainty rejection.

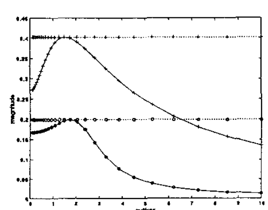

Figure 6: Four kinds of weights used in the example (a) Constant MPR (dashed line marked by '+'), (b) Dynamic MPR (solid line marked by '+'), (c) Constant CPR (dashed line marked by 'o'), (d) Dynamic CPR (solid line marked by 'o')

5. Conclusion

The main contribution of this work is the development of a systematic method of choosing the exact weights which allows for interval plants to be robustly controlled using the $\mathcal{H}_\infty$ synthesis tools. By using the CPR, the results of this paper provide a method for weight selection in

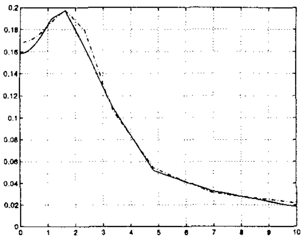

Figure 7: Critical Perturbation Radius (solid line) and its Approximated Function (dash line)

Weights: $\epsilon = 2.83$				
	r_{MPR}	$W_{MPR}(s)$	r_{CPR}	W_{CPR}
$Magnitude$	0.395	$Eq.(28)$	0.2	$Eq.(29)$
$\|T_{zw}(s)\|_\infty$	1	0.8	0.7	0.4

Table 1: Frequency response of different weights

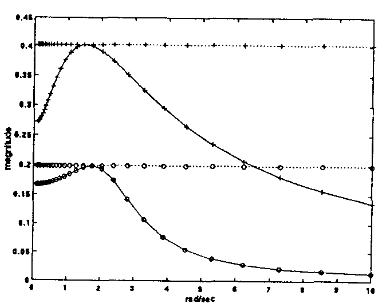

Figure 8: Frequency response of T_{zw} using constant MPR (marked by '+'), dynamic MPR (marked by 'x'), constant CPR (marked by 'o'), and dynamic CPR (marked by '*') weights

$\mathcal{H}_\infty$ design, and the uncertain interval plant can be modeled by the well-developed $\mathcal{H}_\infty$ analysis and framework.

Further research is continuing in the development of an iterative $\mathcal{H}_\infty$ design procedure for interval plants. This procedure will involve an iterative design methodology with alternate weight selection using CPR analysis, followed by $\mathcal{H}_\infty$ controller synthesis.

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