Exact Nyquist-like Stability Results for Ellipsoidal Uncertainties

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Abstract— In this paper we develop a stability criterion for systems with uncertainties which are manifested in the frequency domain by simply-connected and closed, arbitrary uncertainty regions which satisfy a mild convexity constraint. In particular, well-known stability results for the case of disk-bounded frequency domain uncertainties are recovered as a special case of the proposed approach. The main results hinge on the definition of the critical direction as the direction of the line the joining the $-1 + j0$ point to the the nominal frequency response at a particular frequency. It is argued that the worst case uncertainties must lie along this line and this idea is exploited to yield a general stability criterion. An example arising from system and uncertainty identification is presented to illustrate the ideas developed in the paper. An application of the results of this paper yields exact and explicit formulae for the robust stability of systems with ellipsoidal parametric uncertainties.

1 Introduction

The resurgent interest in robustness issues beginning with the decade of the 1980's was initially directed towards a frequency domain characterization of uncertainties [1]. Several very useful results have now emerged which adequately address the analysis question for some important classes of *unstructured* uncertainties. Typically, some frequency domain norm bound on the size of the uncertainty is assumed. For SISO systems this is equivalent to having a disk-bound on the uncertain frequency response:

$$g(j\omega) = g_o(j\omega) + \delta(j\omega), \ \ |\delta(j\omega)| \leq r(\omega)$$

The MIMO equivalent typically uses the induced matrix 2-norm to get

$$G(j\omega) = G_o(j\omega) + \Delta(j\omega), \|\Delta(j\omega)\|_2 \leq r(j\omega)$$

The mathematical problems associated with the analysis of robustness for these classes of unstructured uncertainties have proven to be remarkably simple, yielding the following necessary and sufficient stability condition:

$$r(\omega) \leq \frac{1}{\|(I + G_o(j\omega))^{-1})\|_2} \ \forall \ \omega$$

It is usually the case, however, that much more information is available about the sources and nature of the uncertainties present in a system. Thus there has recently been greater interest in various classes of highly structured uncertainties, involving parametric and non-parametric uncertainties. The question of robust stability in the presence of this type of structured uncertainty can be attacked using several approaches currently available. One common approach is to transform the problem into an MIMO feedback structure where the uncertain components are all assembled into a diagonal uncertainty feedback matrix, Δ, with a forward-path MIMO transfer matrix M(s). The stability problem is then reduced to determining conditions on the uncertain parameters under which a determinantal condition of the form

$$Det[I - M\Delta] \neq 0 \tag{1}$$

is satisfied.

Gaston and Safonov [2] give a useful but computationally demanding iterative algorithm to obtain an exact solution to this problem in the form of the stability margin, k_m. Doyle and his collaborators [3],[4],[5] have also used condition (1) to define the robustness measure μ (which is proportional to the inverse of the stability margin k_m) and have derived upper and lower bounds for the much more general block-diagonal uncertainty problem. The upper bounds on μ are particularly useful since these can be used to obtain sufficient conditions for robust stability, while the lower bounds give some indication of the degree of conservatism involved. If the uncertainties in the matrix Δ (MIMO blocks or scalars) are unconstrained in phase, the upper bounds computed using an optimal similarity transformation matrix which commutes with the structure of Δ are particularly easy to obtain and are known to be exact in many instances. The situation is significantly more difficult when the uncertainties are real, and to date, robustness estimates are only available from these methods by computing upper and lower bounds.

A motivation for the present paper has been the aim of developing a conceptual analytic framework for attempting to solve robustness problems which involve highly structured, real parametric uncertainties, *directly* in the (Nyquist) frequency domain. The first step in this regard is directed at SISO systems characterized by m real, uncertain parameters, $p \in \mathcal{R}^m$, as follows:

$$g(s) = g(s,p) = g_o(s) + \delta(s), \quad g_o(s) = g(s,p_o) \quad (2)$$

where $g_o(s)$ and p_o respectively denote the nominal transfer function and the nominal parameter set, and $\delta(s)$ is an additive uncertainty.

The key idea in the analysis for such SISO systems would be to obtain the nominal polar plot of $g_o(j\omega)$, and then to map the parameter space perturbations, $p - p_o = \delta_p \in \mathcal{D}_p$ into the frequency domain as uncertainty regions about the nominal frequency response. A challenging aspect of this process, of course, is precisely the problem of obtaining the uncertainty templates via a parameter space to frequency domain map[1]. In general, the frequency domain uncertainty templates resulting from parametric uncertainties in transfer functions give rise to irregularly shaped templates, and some encouraging work ([6], [7]) has been done in this area.

Instead of working with the irregularly shaped uncertainty templates, one could essentially revert to an unstructured SISO analysis by using a circle of appropriate radius and center to circumscribe the uncertainties. However, the results would be unnecessarily conservative, since we would be ignoring the known information about the uncertainties and would in fact be including uncertainties which will never arise in the given system.

In this paper we propose a frequency domain stability criterion which provides a conceptual framework for obtaining a non-conservative answer to the robust stability question. The result will enable us to work directly with the frequency response information, with no need to augment the system to isolate each uncertain parameter. Furthermore we shall show that it is often possible to derive explicit closed form solutions for the proposed robustness measure.

2 The Critical Direction: A Neoclassical Approach to Stability Margins

Consider a SISO system represented by

$$g(s) = g_o(s) + \delta(s) \quad (3)$$

where $g_o(s)$ is the exactly known nominal model of an open-loop system and $\delta(s)$ denotes an additive uncertainty or error. Uncertainties may also appear as multiplicative perturbations of the nominal system, or as parametric variations on pole/zero locations and/or variations of the coefficients of the numerator and denominator polynomials in the transfer functions. However we shall use the additive perturbation model for ease of exposition, with the other cases covered by explicit reference to the nominal and uncertain system, $g_o(s)$ and $g(s)$, respectively.

Let us make the assumptions that:

1. The nominal system $g_o(s)$ is stable under unity negative feedback.
2. $g_o(s)$ and $g(s)$ have the same number of open-loop unstable poles

Under these assumptions, the unity feedback system will become unstable if and only if the number of encirclements of the $-1 + j0$ point by the Nyquist diagram changes as a result of the perturbation $\delta(s)$, namely, if and only if the nominal Nyquist diagram is so perturbed by the effect of $\delta(s)$ as to pass through the critical point $-1 + j0$.

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[1] Frequency domain uncertainty templates are sometimes described in the literature ([7]) as *value sets.*

Figure 1 shows a typical Nyquist diagram with an irregular uncertainty region at a particular frequency. In order to facilitate the exposition to follow we will now define the following entities, identified on Figure 1, which will be useful in the subsequent development:

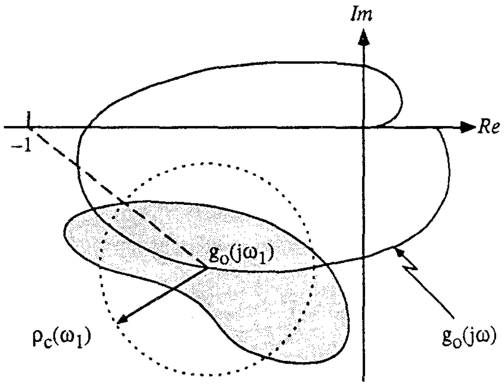

Figure 1: The critical direction and critical perturbation radius.

1. The *nominal* frequency response, $g_o(j\omega)$
2. The *critical direction,* $d(j\omega)$

   $$d(j\omega) = -\frac{1 + g_o(j\omega)}{|1 + g_o(j\omega)|} \tag{4}$$

   which may be interpreted as the unit vector in the direction of the line joining the critical point $-1 + j0$ to the point $g_o(j\omega)$ on the nominal Nyquist diagram at the frequency ω. The critical direction $d(j\omega)$ will prove to be a pivotal element in the theoretical development to follow.
3. The *class of critical perturbations,* δ_c

   $$\delta_c = \{\delta_c(s) = g(s) - g_o(s) \mid \delta_c(j\omega) = \alpha d(\omega), \quad \alpha \in \Re_+\} \tag{5}$$

   namely, the set of perturbations with frequency response lying along the critical direction $d(\omega)$.
4. The *critical perturbation radius,* $\rho_c(\omega)$

   $$\rho_c(\omega) = \max_{\alpha \in \Re_+} \{\alpha \mid \alpha d(j\omega) \in \delta_c\} \tag{6}$$

Finally, we also define the *infinity norm* of the transfer function $g(s)$:

$$\|g\|_\infty := \sup_\omega |g(j\omega)| \tag{7}$$

With these definitions in hand we can now proceed to state the following robust stability criterion:

Theorem 1. Consider the system $g(s) = g_o(s) + \delta(s)$ and suppose that the nominal unity feedback system is stable and that $g(s)$ and $g_o(s)$ have the same number of open-loop unstable poles. Then, if the class of critical perturbations δ_c is a convex set, the uncertain system will remain stable if and only if

$$\frac{\rho_c(\omega)}{|1 + g_o(j\omega)|} < 1 \quad \forall \; \omega$$

or equivalently, if and only if

$$\left\| \frac{\rho_c(s)}{1 + g_o(s)} \right\|_\infty < 1 \tag{8}$$

Proof: Under the given assumptions, it follows from the classical Nyquist stability criterion that the uncertain system will remain stable if and only if the perturbed Nyquist diagram does not contain the -1+j0 point, namely if and only if

$$g(j\omega) = g_o(j\omega) + \delta(j\omega) \;\neq\; -1 \;\forall\; \omega \tag{9}$$

or equivalently, if and only if

$$1 + g_o(j\omega) + \delta(j\omega) \neq 0 \quad \forall \quad \omega \tag{10}$$

At this point we note that from (5), (6) and (10), it suffices to consider only perturbation subsets which lie along the critical direction, namely perturbations for which

$$\delta(j\omega) = \alpha d(j\omega), \;\; 0 \leq \alpha \leq \rho_c(\omega) \tag{11}$$

because these are the only perturbations which could cause condition (10) to be violated. Thus, for robust stability we have the necessary and sufficient condition

$$\frac{\alpha d(j\omega)}{1 + g_o(j\omega)} \neq -1 \;\forall\; \omega \tag{12}$$

After invoking definition (4), it follows that a sufficient stability condition is given by

$$\frac{\alpha}{|1 + g_o(j\omega)|} < 1 \;\forall\; \omega \tag{13}$$

Furthermore, since $\alpha \leq \rho_c(\omega)$, stability is ensured if

$$\frac{\rho_c(\omega)}{|1 + g_o(j\omega)|} < 1 \;\forall\; \omega \tag{14}$$

or

$$\|\frac{\rho_c(s)}{1 + g_o(s)}\|_\infty < 1 \tag{15}$$

The proof is completed by noting that conditions (13) - (15) are also *necessary* for stability. Indeed, invoking continuity arguments for simply connected, closed uncertainty contours we claim that if condition (15) is violated, there will always be a perturbation in the allowable set which leads to instability. To prove this claim, suppose

$$\frac{\rho_c(\omega)}{|1+g_o(j\omega)|} = \frac{1}{\gamma} > 1 for\ some\ \omega = \omega_o \qquad (16)$$

and recall that the class of possible destabilizing or critical perturbations is assumed to be convex and is characterized by equation (5), with $0 \leq \alpha \leq \rho_c(\omega_o)$. If condition (16) holds we have

$$\rho_c(\omega_o) = \frac{1}{\gamma}|1+g_o(j\omega_o)| > |1+g_o(j\omega_o)|$$

We can thus choose $\alpha = |1+g_o(j\omega_o)|$ to get the admissible uncertainty

$$\delta_c(j\omega_o) = \alpha d(j\omega_o) = -(1+g_o(j\omega_o))$$

which violates (10) and hence implies instability. This completes the proof.

3 Discussion: A Unified Generalized Theory

The result in Theorem 1 accommodates highly structured perturbations with fairly general frequency response characteristics, bounded not only in magnitude, but also constrained in phase. The assumption that the uncertainty region is simply connected and a continuous function of the uncertain parameters holds generically for real systems. Moreover the condition requiring convexity of the class of critical perturbationin δ_c is a technical requirement for the proof as presented. Clearly the convexity condition is trivially satisfied when the straight line $\delta_c(j\omega)$ is a continuous function of α. A stronger condition for which equation (8) would also constitute a necessary and sufficient stability condition is to insist that the entire uncertainty region be convex. Obviously, the convexity requirement on the subset δ_c is a much weaker condition, and even this can be further relaxed.

In the rest of this section several arguments are given to show how Theorem 1 provides connections with both the classical as well as more modern robustness results. It is to be noted that Theorem 1 and the comments below apply equally to continuous-time and discrete-time systems.

3.1 The critical point vs the critical line

The main idea exploited in Theorem 1 is the notion of the critical direction associated with the return difference $f(j\omega) = 1+g_o(j\omega)$. The return difference $f(j\omega)$ has appeared in all the classical and more recent robust control literature for both SISO and MIMO systems, most notably as the denominator of the sensitivity and complementary sensitivity functions. However, in general attention has been focussed primarily on the magnitude of $f(j\omega)$, since only the magnitude appears in the final results obtained from the popular analyses associated with unstructured uncertainties. It is the opinion of the authors that substantial advantages can be gained by considering not only the magnitude of the return difference transfer function, but also its associated direction, defined in this paper to be the critical direction $d(j\omega)$. For example Theorem 1 asserts that the only uncertainties relevant to stability analysis are those which lie along the straight line defined by $d(j\omega)$.

Another point of unification and generalization is the suggestion that the notion of the critical phenomena having to do with robust stability should include not only a critical *point*, but also a critical *line*. For whereas the popular classical gain and phase margins can be obtained by concentrating attention on a *critical point*, from which maximal tolerable linear and rotational distances are computed to get gain and phase margins respectively, Theorem 1 motivates very strongly the advantages which are gained by a consideration of the *critical line* characterized by the *critical point* and the *critical direction.*

3.2 Connections with disk-bounded uncertainties

The result of Theorem 1 also recovers as a special case the simple unstructured uncertainty whose frequency response is bounded only in magnitude, but otherwise unconstrained, namely the case of circular, or more precisely, disk bounded uncertainties. Indeed it is easy to see that for disk uncertainties, the critical perturbation radius is precisely equal to the radius of the uncertainty disk, i.e. the phase independent magnitude bound on the uncertainty. Note that for disk-bounded uncertainties, both the uncertainty region and δ_c are convex sets.

It should be noted, however, that although circular frequency domain uncertainty regions lead to a simpler mathematical problem, the associated results may be unduly conservative. To illustrate this fact, let us assume that at a particular frequency the effect of the perturbation $\delta(s)$ is such that the frequency response of the system $g(s) = g_o(s) + \delta(s)$ is as shown in Figure 1. Then Theorem 1 stipulates that for stability, the maximal excursion *along the critical direction*, of $g(j\omega)$ from $g_o(j\omega)$, should be less than $|1 + g_o(j\omega)|$, the distance of the nominal frequency response $g_o(j\omega)$ from the -1+j0 point. For the uncertainty shown in Figure 1 the result of Theorem 1 constitutes an exact stability condition. Other approaches to this problem generally yields conservative results, since the structure of the problem is ignored and an essentially unstructured representation is used. This will certainly be the case if the uncertainty region at each frequency is completely circumscribed by a circle of large enough radius or if the bands swept by $g_o(s) + \delta(s)$ as s traverses the imaginary axis, are considered and circles of appropriate radii and with centers inscribed within such bands. In this latter case, the inscribed circles may have no physical interpretation, for example the centers of the circles may no longer correspond to the nominal frequency response, and hence the problem structure would again be lost.

On the other hand, the tools which have been developed using unstructured disk bounded uncertainties can still be used while still preserving necessity and sufficiency in the stability result. To accomplish this goal, Theorem 1 stipulates that at each frequency the uncertainty disks should have a radius given by $\rho_c(\omega)$. If uncertainty disks are thus defined, then the unstructured approach would yield the same result as that given in Theorem 1, and would hence be necessary and sufficient. Note that this fact may appear somewhat counter-intuitive since these uncertainty disks do not necessarily circumscribe the entire uncertainty region at each frequency. In some sense, the proposed disk bounds exclude exactly those perturbations which are irrelevant to the stability analysis and thus attains necessity and sufficiency.

3.3 Implications for robustness analysis

Many H_∞ design problems invoke the robustness constraint

$$\|W(s)S(s)\|_\infty < 1$$

where $W(j\omega)$ is a weighting function and $S(j\omega)$ is the sensitivity function $S(j\omega) = \frac{1}{1+g_o(j\omega)}$.

A major engineering challenge has been the selection of appropriate functional forms for the weights $W(s)$. In this context, it is asserted that the specification

$$|W(j\omega)| = \rho_c(\omega)$$

will provide an exact weight for H_∞ robustness analysis in the presence of this class of structured uncertainties. To see this, note that the robustness measure becomes

$$\left\| \frac{\rho_c(s)}{1 + g_o(s)} \right\|_\infty < 1$$

This latter problem may now be attacked by invoking now standard H_∞ analysis methods.

A key feature of the emerging solutions to the H_∞ methods is their implementation in terms of the state space representations of the plant and of the weighting functions. Thus it is of some interest, not simply to specify some magnitude bounds for the frequency response of the weighting functions, but also to obtain proper state space realizations of the associated transfer functions. A first step in this direction is to obtain explicit closed-form solutions for the weighting function $W(s) = \rho_c(s)$ for which state space realizations can then more readily be derived.

In the following section we present an example of a system and uncertainty model derived from statistical parameter estimation techniques. The resulting ellipsoidal parameter-space uncertainty model, in addition to its intrinsic merits, facilitate a direct frequency response robustness analysis and an explicit characterization of $\rho_c(\omega)$, thus illustrating the material presented thus far in this paper.

4 Model Estimation and Ellipsoidal Parametric Uncertainties: A Frequency Domain Interpretation

Let us consider a general system model given by the discrete-time transfer function

$$H(z) = \sum_{k=1}^{q} h_i z^{-k} \tag{17}$$

with q parameters defined by $\mathbf{h} = [h_1, h_2, ..., h_q]^T$. The nominal system, denoted $H^o(z)$ is obtained when $\mathbf{h}$ assumes the nominal values $\mathbf{h}^o = [h_1^o, h_2^0, ..., h_q^o]^T$ and the real system is modeled as $H(z) = H^o(z) + \delta H(z)$. As discussed in the previous section, our interest will be directed towards determining bounds on the size of $\delta H(e^{j\omega})$ as the system parameters vary within some set, and then use these uncertainty bounds to formulate exact robust-stability conditions for the unity feedback loop.

Let $\mathbf{h} \in \mathcal{R}^q$ be such that $\mathbf{h} = \mathbf{h}^o + \delta\mathbf{h}$. Then we define an ellipsoidal parameter space as follows:

$$\delta\mathbf{h} = \mathbf{h} - \mathbf{h^o}, \;\; \delta\mathbf{h} \in \mathcal{E}_h \qquad (18)$$

$$\mathcal{E}_h := \left\{\delta\mathbf{h}^T \mathbf{Q}_h^{-1} \delta\mathbf{h} \le 1; \;\; \mathbf{Q}_h = \mathbf{Q}_h^T > 0\right\} \qquad (19)$$

The set $\mathcal{E}_h$ constitutes an ellipsoidal description of the uncertainty. We argue that, in addition to the remarkable mathematical tractability which we shall show later, the ellipsoidal parametric description of uncertainties is quite natural in many application and offers, in contrast to hyper-rectangular descriptions, the further advantage of allowing the dependence among various system parameters to be taken into explicit account. Ellipsoidal models often arise quite naturally, as for example whenever linear regression or least squares analysis is used in model estimation. (See for example ([9])).

Now our main interest is in finding the map from the parameter space to the frequency domain which would then enable us to invoke the robustness results obtained earlier. Clearly the nominal parameters $\mathbf{h^o}$ will map to the nominal transfer function $H^o(z) = \sum_{k=1}^{q} h_k^0 z^{-k}$. Consider now the frequency response $H(e^{j\omega})$ where $H(z) = \sum_{k=1}^{q} h_k z^{-k}$ contains parameters $\mathbf{h} = [h_1, h_2, ..., h_q]^T$ lying in the ellipsoid derived earlier. Under these conditions, the next lemma shows that the parameter space ellipsoid maps precisely to ellipses at all frequencies except for $\omega = 0$ and $\omega = \pi$.

Lemma 2. The parameter space ellipsoid defined by

$$(\mathbf{h} - \mathbf{h}^o)^T \mathbf{Q}_h^{-1} (\mathbf{h} - \mathbf{h}^o) \le 1$$

maps to frequency domain ellipses defined by

$$\left(\overline{\mathbf{H}}(e^{j\omega}) - \overline{\mathbf{H}}^0(e^{j\omega})\right)^T \mathbf{Q}_\omega^{-1} \left(\overline{\mathbf{H}}(e^{j\omega}) - \overline{\mathbf{H}}^0(e^{j\omega})\right) \le 1 \qquad (20)$$

where

$$\overline{\mathbf{H}}(e^{j\omega}) = \begin{bmatrix} \Re H(e^{j\omega}) \\ \Im H(e^{j\omega}) \end{bmatrix} = \mathbf{W}(\omega)\mathbf{h} \in \Re^2$$

$$\mathbf{Q}_\omega := \mathbf{W}(\omega)\mathbf{Q}_h\mathbf{W}(\omega)^T \in \Re^{2\times 2}$$

and

$$\mathbf{W}(\omega) = \begin{bmatrix} cos\omega & cos2\omega & . & . & . & cosq\omega \\ sin\omega & sin2\omega & . & . & . & sinq\omega \end{bmatrix}$$

At $\omega = 0$ and $\omega = \pi$, $\mathbf{W}(\omega)$ is non-singular and the frequency response uncertainty is entirely real and given by

$$|H(j\omega) - H^o(j\omega)| \le \sqrt{\mathbf{v}_R(\omega)\mathbf{Q}_h\mathbf{v}_R^T(\omega)} \qquad (21)$$

where $\mathbf{v}_R(\omega) = [cos\omega \;\; cos2\omega \;\; . \; . \; . \;\; cosq\omega]$

Proof: The proof makes use of a result given in ([9]). Details are omitted.

Equation (20) indicates that the frequency domain uncertainity templetes $\delta H(e^{j\omega})$ are described by ellipses. Equipped with the analytic expression for the frequency domain elliptical uncertainty bounds, we can now proceed to state the main robust stability result for ellipsoidal uncertainties.

Theorem 2. Under the assumptions of nominal closed-loop stability and that the nominal perturbed system share the same number of open-loop unstable poles, the unity negative feedback system with open-loop transfer function (17) and uncertain parameters (18), (19) remains stable if and only if

$$\|\eta(z)\|_\infty < 1$$

where

$$|\eta(e^{j\omega})| = \frac{1}{\mathbf{d}_c^T(\omega)\, \mathbf{Q}_\omega^{-1}\, \mathbf{d}_c(\omega)}, \quad \omega \in (0, \pi) \qquad (22)$$

and

$$|\eta(e^{j\omega})| = \frac{\mathbf{v}_R(\omega)\, \mathbf{Q}_h\, \mathbf{v}_R^T(\omega)}{\mathbf{d}_c^T(\omega)\, \mathbf{d}_c(\omega)}, \quad \omega = 0, \pi \qquad (23)$$

where

$$\mathbf{d}_c(\omega) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \mathbf{W}(w)\mathbf{h}^o$$

Proof. Consider the case $\omega \in (0, \pi)$. Using two-dimensional vector analysis and the definitions in Section 2 it becomes clear that $\mathbf{d_c}(\omega)$ as given in Theorem 2 corresponds precisely to the unnormalized critical direction $d(j\omega)$ defined earlier. Now from Lemma 2, the frequency response $H(e^{j\omega})$

at each frequency is located inside an ellipse with center $H^o(e^{j\omega})$. Thus stability will be guaranteed if, at all frequencies, the following relationship holds:

$$\rho_c(\omega) < \sqrt{\mathbf{d}_c^T(\omega)\mathbf{d}_c(\omega)} = |1 + H^o(e^{j\omega})|$$

In other words, stability is ensured if the critical perturbation radius $\rho_c(\omega)$, namely, the distance from the center of the ellipse to the point at which the vector $\mathbf{d}_c(\omega)$ intersects with the ellipse, is always less than the length of the vector $\mathbf{d}_c(\omega)$. A fundamental result from two-dimensional coordinate geometry gives the length of the line joining the center of an ellipse to the point of intersection along the vector $\mathbf{d}_c(\omega)$ as

$$\rho_c(\omega) = \frac{\sqrt{\mathbf{d}_c^T(\omega)\,\mathbf{d}_c(\omega)}}{\sqrt{\mathbf{d}_c^T(\omega)\,\mathbf{Q}_\omega^{-1}\,\mathbf{d}_c(\omega)}}$$

The proof is completed by invoking Theorem 1 with $\rho_c(\omega)$ as given above, and using the definition of the $\|.\|_\infty$-norm. The proof for the singular cases where $\omega = 0, \pi$ is analogous.

5 Conclusion

In this paper we have developed a stability criterion for systems with uncertainties which are manifested in the frequency domain by simply-connected and closed, but otherwise arbitrary regions about the nominal system. In particular, well-known stability results for the case of disk-bounded frequency domain uncertainties are recovered as a special case of the proposed approach. The main results hinge on the definition of the critical direction as the direction of the line joining the nominal frequency response at a particular frequency, to the -1+j0 point. It was argued that the worst case uncertainties must lie along this line and this idea is exploited to yield a general stability criterion. An example arising from system and uncertainty identification was presented to illustrate the ideas developed in the paper. An application of the results of this paper yields exact and explicit formulae for the robust stability of systems with ellipsoidal parametric uncertainties.

Further work is presently underway to develop an interactive design method using the parameter estimation in conjunction with the H_∞ design. Additionally, the critical direction based worst-case perturbation parametrization is also being extended to the case of MIMO block structured uncertainties.

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