

THE NYQUIST ROBUST STABILITY MARGIN—A NEW METRIC FOR THE STABILITY OF UNCERTAIN SYSTEMS

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SUMMARY

The Nyquist robust stability margin k_N is proposed as a new tool for analysing the robustness of uncertain systems. The analysis is done using Nyquist arguments involving eigenvalues instead of singular values, and yields exact necessary and sufficient conditions for robust stability. The concept of a critical line on the Nyquist plane is defined and used to calculate a critical perturbation radius which in turn is used to produce k_N . The new approach gives alternatives to computing exact stability margins in some cases of highly directional uncertainty templates where other models are not applicable.

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1. INTRODUCTION

Many of the now well-known results on robust stability can trace their origins to the Nyquist stability criterion (SISO case), or the generalized Nyquist stability criterion (MIMO case).¹ Such is the case for the well-known multivariable stability margin k_m ² and the structured singular value μ .³ Although the generalized Nyquist criterion addresses the issue of stability using complex variable (eigenvalue or transfer function) arguments, the predominant robustness analysis methods typically rely on arguments based on real singular value or magnitude bounds. One reason for this choice is clearly the more advantageous conditioning of singular values for numerical calculations. However another important factor is the historical fact that uncertainty measures were initially proposed for MIMO systems using singular-value bounds, which collapse to magnitude bounds (disks) for SISO systems.

In this paper we introduce the concept of the *Nyquist robust-stability margin*, k_N , for characterizing the closed-loop stability of uncertain systems. The approach makes direct use of Nyquist arguments and is based on the analysis of the uncertainty characterization directly on the Nyquist plane, hence avoiding undue conservatism that may result through the use of singular-value or magnitude upper bounds.

A key element in the new approach is the *critical direction theory* applied to uncertainties in the Nyquist plane. A basic version of the critical direction concept was first presented in Reference 5 as a tool for analysing the robust stability of polynomials with ellipsoidal uncertainties. In Reference 6 the critical direction notion is redefined directly on the Nyquist plane to facilitate the derivation of necessary and sufficient robust stability conditions for transfer functions subject to arbitrary perturbations. In this paper we further extend the critical direction theory for the

analysis of MIMO systems by formulating the concept in the generalized Nyquist plane, introducing new nomenclature to facilitate the seamless extension from the SISO to the MIMO case, and defining the new concept of the Nyquist robust stability margin.

The critical direction method is based on recognizing that, at any given frequency on the Nyquist plane, there is only one direction of perturbation of relevance to the stability analysis. This critical direction is defined by the oriented line that has its origin at the location of the nominal eigenvalue or transfer function of the unperturbed system and passes through the critical point $-1 + j0$. In fact, from a stability point of view, all points on the uncertainty eigentemplate that do not lie on the critical line can be ignored. This allows the characterization of robust stability margins for uncertain systems characterized by irregular perturbation templates, a problem that poses significant challenges to other analysis methods. Other attractive features of the Nyquist robust-stability margin theory are that its MIMO form is a natural extension of the SISO case, and that its versions for continuous and discrete systems are formally identical.

The paper is organized as follows. Section 2 gives a succinct review of robust-stability margin results for the SISO and MIMO cases. Section 3 formulates the critical direction theory for analysing the robustness of uncertain SISO systems, and Section 4 gives the details of the new critical direction theory for the MIMO case. A SISO and a MIMO example are given in Section 5, followed by final remarks and conclusions in Section 6.

2. FREQUENCY-DOMAIN APPROACH TO ROBUST STABILITY ANALYSIS

The new Nyquist robust stability margin is an object derived using Nyquist arguments. In order to provide a contextual background for the ensuing discussions, in this section we review relevant SISO and MIMO results on robust stability analysis, placing particular attention on the role of Nyquist arguments that also lie at the root of the new developments presented here.

The transfer function $g(s)$ of a SISO system can be written in terms of an additive perturbation $\delta(s)$ about a nominal transfer function $g_o(s)$

$$g(s) = g_o(s) + \delta(s), \delta(s) \in \mathbf{d} \quad (1)$$

where \mathbf{d} represents the set of allowed perturbations. When all $\delta(s) \in \mathbf{d}$ are considered, then at each frequency the map $g(j\omega) = g_o(j\omega) + \delta(j\omega)$, defines a region denoted *uncertainty template* or value set.⁷ It is then possible to visualize the transfer-function uncertainty directly in terms of the classical Nyquist plot as uncertainty templates about the locus of the nominal system $g_o(j\omega)$.

The mathematical description for the set of perturbations may be formulated in different spaces. For example, the Laplace-function domain \mathbf{d} is often defined implicitly in terms of a frequency-domain description. Such is the case where the uncertainty templates are specified to be circular at each frequency. This makes the analysis of robust stability particularly straightforward. On the other hand, the templates for a number of useful uncertainty descriptions are not circular but rather irregularly shaped, exhibiting highly directional features. This is the case for parametric uncertainty descriptions of the form $\delta(s) = g(s, \mathbf{p}) - g(s, \mathbf{p}^0)$, where $g(s, \mathbf{p}^0)$ and \mathbf{p}^0 respectively denote the nominal transfer function and the nominal parameter set, and where $g(s, \mathbf{p})$ is characterized by m real, uncertain parameters $\mathbf{p} \in \mathbf{d}_p \subset \mathcal{R}^m$. In this framework the set \mathbf{d} is defined implicitly through the Euclidean domain \mathbf{d}_p . Typical uncertainty representations for the parameter uncertainty set \mathbf{d}_p are polytopes and ellipsoids. The special case where the uncertainty templates are ellipses is an interesting example of directional uncertainty templates which also has a tractable mathematical representation.

The representation (1) is quite general and adequately encompasses additive perturbations as well as uncertainties that may appear as multiplicative perturbations of the nominal system,

parametric variations on pole/zero locations, and/or variations of the coefficients of the numerator and denominator polynomials in the transfer function. The difference in each case is the shape and orientation of the uncertainty templates associated with $\delta(s)$.

Of particular interest are three types of SISO uncertainty descriptions. Specifically, we denote the set of uncertainties giving rise to *unstructured* (circular), *elliptical* and *structured* (arbitrary) templates, respectively by the notation \mathbf{d}_U , \mathbf{d}_E , and \mathbf{d}_S .

In the case of uncertain MIMO systems, the matrix transfer function $G(s)$ can be written as an additive perturbation $\Delta(s)$ about a known nominal transfer matrix $G_o(s)$

$$G(s) = G_o(s) + \Delta(s), \Delta(s) \in \mathcal{D} \quad (2)$$

where \mathcal{D} is the set of allowed perturbations. The uncertainty set \mathcal{D} is described in terms of available information about the modeling errors. We use the designation \mathcal{D}_U to represent *unstructured* uncertainties with a single norm-bound, \mathcal{D}_B to denote *block-diagonal norm-bounded uncertainties*, \mathcal{D}_C to represent *element-wise circular (disk-bounded) uncertainties* and \mathcal{D}_E to represent *element-by-element elliptical* uncertainties. Other uncertainty descriptions are defined in terms of errors in the elements of the state-space matrices, or errors in the real parameters of the matrix transfer functions. In all these cases the uncertainty description can be mapped (via eigenvalue inclusion regions) onto frequency-response uncertainty templates in the Nyquist plane, which then form the basis for defining the Nyquist robust stability margin.

2.1. Classical SISO robust-stability results

The SISO robust stability analysis is concerned with the stability of the system that results when the system $g(s)$ in (1) is arranged in a unity negative-feedback configuration. It is normally assumed that (i) the nominal system is stable under unity negative-feedback, and that (ii) the nominal and uncertain system have the same number of open-loop unstable poles. These assumptions are adopted throughout this paper.

The most studied case is perhaps the case in which the uncertainty description is unstructured, $\delta(s) \in \mathbf{d}_U$ i.e., when $|\delta(j\omega)| \leq |W(j\omega)|$, where $W(s)$ is a known function whose frequency-response magnitude defines the radius of the circular uncertainty templates.

Invoking the Nyquist stability criterion leads to the necessary and sufficient stability requirement that all uncertainty templates exclude the critical point $-1 + j0$, i.e.,

$$g_o(j\omega) + \delta(j\omega) \neq -1 \quad \forall \delta(j\omega) \text{ and } \forall \omega \quad (3)$$

This condition can be expressed through the inequality⁸

$$\left| \frac{W(j\omega)}{1 + g_o(j\omega)} \right| < 1 \quad \forall \omega$$

or equivalently,

$$\left\| \frac{W(s)}{1 + g_o(s)} \right\|_{\infty} < 1 \quad (4)$$

The robustness analysis for uncertainty descriptions with non-circular templates is more challenging. For this reason it is quite common for elliptical or arbitrarily shaped uncertainty templates, arising for example from parametric uncertainties, to be circumscribed by an appropriate circle.⁹ Although this approach yields sufficient conditions for robust stability, it is nevertheless inherently conservative.

2.2. Classical MIMO robust-stability results

As in the SISO case, the robustness analysis for MIMO systems makes use of the generalized Nyquist stability criterion,¹ where avoidance of the critical point $-1 + j0$ is also at the centre of interest from the point of view of absolute stability assessment. This has been the major focus of much research interest in the development of robust multivariable stability margins. Consider the uncertain transfer matrix $G(s)$ in (2) with nominal model $G_o(s)$ affected by an uncertainty $\Delta(s)$. Again it is assumed that (i) the nominal system is stable under unity negative-feedback, and that (ii) the nominal and uncertain system have the same number of open-loop unstable poles.

The conditions under which no eigenvalue of the uncertain system $G(j\omega) = G_o(j\omega) + \Delta(j\omega)$ is equal to $-1 + j0$, i.e.,

$$\lambda(G_o(j\omega) + \Delta(j\omega)) \neq -1 \quad \forall \Delta(j\omega) \text{ and } \forall \omega \quad (5)$$

is readily shown to lead to the determinantal condition

$$\det(I + M(j\omega)\Delta(j\omega)) \neq 0 \quad \forall \Delta(j\omega) \text{ and } \forall \omega \quad (6)$$

where $M(j\omega) := (I + G_o(j\omega))^{-1}$. Condition (6) is of course equivalent to the eigenvalue condition

$$\lambda(M(j\omega)\Delta(j\omega)) \neq -1 \quad \forall \Delta(j\omega) \text{ and } \forall \omega \quad (7)$$

Note that the robust stability condition (5) is obtained by invoking the generalized Nyquist stability criterion. The multivariable stability margin k_m and the structured singular value μ were independently defined in References 2 and 3, respectively, in terms of the determinantal stability condition (6), rather than the eigenvalue conditions (5) or (7). For an uncertainty of a given description \mathcal{D} , the multivariable stability margin k_m may be defined as the matrix 2-norm of the smallest destabilizing uncertainty in the given class, namely

$$k_m(\omega) = \min_{\Delta \in \mathcal{D}} \{ \bar{\sigma}(\Delta(j\omega)) : \det(I + M(j\omega)\Delta(j\omega)) = 0 \} \quad (8)$$

This definition corresponds to the reciprocal of the robustness measure μ defined in Reference 3, which is defined as the inverse of the norm of the smallest destabilizing uncertainty in the class, that is

$$\mu(\omega) = \left\{ \min_{\Delta \in \mathcal{D}} \{ \bar{\sigma}(\Delta(j\omega)) : \det(I + M(j\omega)\Delta(j\omega)) = 0 \} \right\}^{-1} \quad (9)$$

Note that if no uncertainty in the allowable class destabilizes the system (i.e., makes an eigenvalue of $G_o(j\omega) + \Delta(j\omega)$ equal to -1 at any frequency) then $k_m = \infty$ and $\mu = 1/k_m = 0$. These margins provide a measure of tolerable uncertainty size.

A number of powerful robust-stability results can be obtained in terms of the robust stability margins (8) and (9). Consider for example the following case of interest. An uncertainty description \mathcal{D} is said to be *closed under contraction and rotation* if for any $\Delta(s) \in \mathcal{D}$, then $\gamma e^{j\theta} \Delta(s) \in \mathcal{D}$ for all $0 \leq \gamma \leq 1$ and for all $0 \leq \theta \leq 2\pi$. Starting from the eigenvalue stability condition (7), we can state the following lemma.

Lemma 1

Consider an uncertainty description \mathcal{D} that is closed under contraction and rotation. Then the uncertain system $G(s) = G_o(s) + \Delta(s)$ is stable for all $\Delta(s) \in \mathcal{D}$ if and only if

$$\sup_{\Delta \in \mathcal{D}} \rho(M(j\omega)\Delta(j\omega)) < 1 \quad \forall \omega \quad (10)$$

Proof. Since the eigenvalue condition (7) is necessary and sufficient for robust stability, it is clear that (10) immediately constitutes a sufficient stability condition. That (7) is also necessary is established by contraposition as follows: consider an uncertainty $\Delta_o \in \mathcal{D}$ such that at some frequency, $\rho(M\Delta_o) = 1/\gamma > 1$, then owing to the closure under contraction we can always find an uncertainty $\Delta_1 \in \mathcal{D}$ with $\Delta_1 = \gamma\Delta_o$, $0 < \gamma < 1$, such that $\rho(M\Delta_1) = 1$. Since the phase of the uncertainty class is arbitrary, owing to the closure under rotation we can assign a scalar phase multiplier to Δ_1 to get Δ_2 such that $\lambda(M\Delta_2) = -1$, which of course implies instability. Thus it follows that for the description \mathcal{D} specified in the Lemma, condition (10) is both necessary and sufficient for robust stability. \square

For the classes of uncertainties covered by Lemma 1, the spectral radius condition (10) is in fact equivalent (after appropriate normalizations) to the definitions of μ and k_m , and thus it naturally follows that μ is equal to the left-hand side of (10). Hence the stability criterion of Lemma 1 can be equivalently formulated as

$$k_m(\omega) > 1 \quad \forall \omega \quad (11)$$

or

$$\mu(\omega) < 1 \quad \forall \omega \quad (12)$$

The k_m and μ stability margins serve to provide a valuable characterization of destabilizing uncertainties. Specific procedures and results have been developed for computing these stability margins (or at least good upper and lower bounds) for various useful classes of uncertainties. Additionally, the μ problem formulation has produced an extensive set of results associated with the class of block diagonal bounded uncertainties, and has thus attained wide acceptance for robustness analysis, and more recently, also for controller synthesis using the μ -synthesis method. In all these applications the computation of the stability margin is effected by calculating singular-value upper bounds which give sufficient, and in some cases necessary and sufficient, stability conditions.

Unfortunately, however, the use of singular-value or magnitude bounds in the characterization of destabilizing uncertainties often causes rich structural properties, including phase and directionality, to be ignored. On the other hand, as we show below, significant advantages can be gained by studying the robustness problem directly from an eigenvalue (or transfer function) point of view, especially since necessary and sufficient stability conditions correspond precisely to an eigenvalue or transfer function condition via the Nyquist criterion.

3. THE CRITICAL DIRECTION METHOD FOR SISO SYSTEMS

In this section we present an exact robust stability result for the case of SISO systems using the critical direction theory. We adopt a nomenclature that permits the extension of the critical-direction concepts to the MIMO case. We also introduce the definition of the Nyquist robust stability margin for SISO systems in a form which readily extends to the MIMO case.

Figure 1 shows a typical Nyquist diagram for a SISO system illustrating the *nominal frequency response* $g_o(j\omega)$ and an irregularly shaped uncertainty template. We define the *critical line* at a given frequency ω as the directed line which originates at the nominal point $g_o(j\omega)$ and passes through the critical point $-1 + j0$. The figure is also useful for identifying the entities defined below:

1. The *critical direction*

$$d(j\omega) = -\frac{1 + g_o(j\omega)}{|1 + g_o(j\omega)|} \quad (13)$$

which may be interpreted as the unit vector that defines the direction of the critical line.

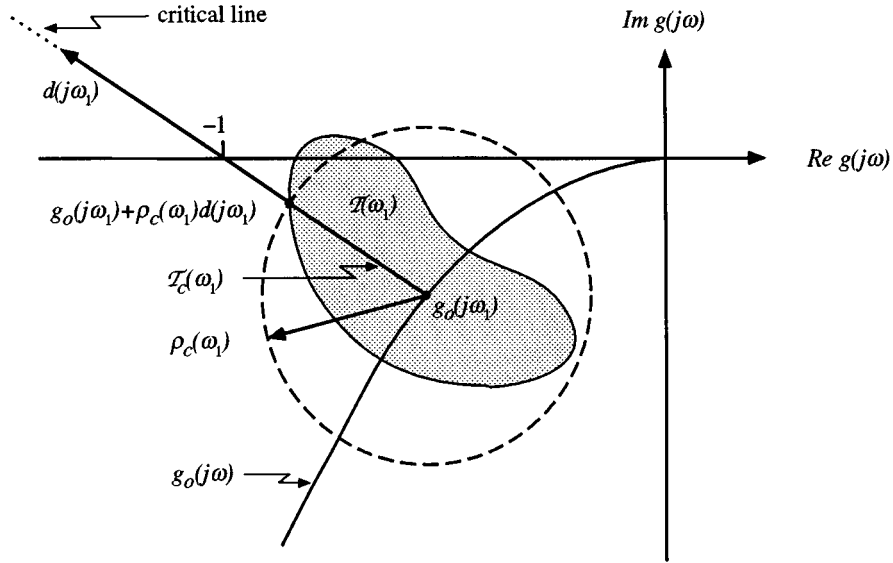


Figure 1. An irregularly shaped uncertainty template at frequency ω_1 (shaded area) and its critical perturbation radius $\rho_c(\omega_1)$. The critical template $\mathcal{T}_c(\omega_1)$ (solid line) is the subset of template points lying on the line segment with end points $g_o(j\omega_1)$ and $g_o(j\omega_1) + \rho_c(\omega_1)d(j\omega_1)$

2. The uncertainty template

$$\mathcal{T}(\omega) = \{g(j\omega) \mid g(j\omega) = g_o(j\omega) + \delta(j\omega), \delta(s) \in \mathbf{d}\} \tag{14}$$

3. The critical template

$$\mathcal{T}_c(\omega) = \{z \in \mathcal{T}(\omega) \mid z = g_o(j\omega) + \alpha d(j\omega) \text{ for some } \alpha \in \mathcal{R}_+\} \tag{15}$$

namely, the set of perturbed frequency-responses lying along the critical direction $d(j\omega)$.

4. The Critical perturbation radius

$$\rho_c(\omega) = \max_{\alpha \in \mathcal{R}_+} \{\alpha \mid z = g_o(j\omega) + \alpha d(j\omega) \in \mathcal{T}_c(\omega)\} \tag{16}$$

As an illustration of the previous definitions note that the critical line at frequency $\omega = \omega_1$ is readily identified in Figure 1 as the directed line with origin at $g_o(j\omega_1)$ and passing through the point $-1 + j0$. The critical direction $d(j\omega_1)$ is simply the unit-length vector that characterizes the direction of the critical line. In addition, the critical radius $\rho_c(\omega_1)$ is shown as the distance between the nominal Nyquist point $g_o(j\omega_1)$ and the point where the critical line intersects with the boundary of the template. Finally, the critical template $\mathcal{T}_c(\omega)$ is also readily characterized in Figure 1 as the subset of the uncertainty template $\mathcal{T}(\omega)$ that intersects with the critical line.

By definition $\mathcal{T}_c(\omega)$ is a subset of the critical line, thus, it follows that $\mathcal{T}_c(\omega)$ is either a single straight-line segment, or the union of such segments. The critical template may also contain isolated points should the boundary of the template $\mathcal{T}(\omega)$ be tangent to the critical direction. Figure 1 shows the case where the critical template is a continuous segment, and hence $\mathcal{T}_c(\omega)$ is a convex set even though the entire template is highly non-convex. The critical template is discontinuous when it is made up of the union of distinct segments. In this case $\mathcal{T}_c(\omega)$ is not convex, but each of its segments is a convex subset.

With these definitions in hand we can proceed to state the following robust-stability theorem. For ease of exposition we assume that the critical template $\mathcal{T}_c(\omega)$ is a convex set for all frequencies. This restriction can be relaxed through obvious modifications to account for each of the convex segments of $\mathcal{T}_c(\omega)$.

Theorem 1

Let the nominal SISO system $g_o(s)$ be subject to an uncertainty $\delta(s) \in \mathbf{d}$. Then the uncertain closed-loop system remains stable under unity feedback if and only if

$$\frac{\rho_c(\omega)}{|1 + g_o(j\omega)|} < 1 \quad \forall \omega \quad (17)$$

The proof of the theorem is omitted since all details are given in Reference 6. Motivated by (17) we now propose the following definition of the *Nyquist robust-stability margin* for SISO systems:

$$k_N(\omega) := \frac{\rho_c(\omega)}{|1 + g_o(j\omega)|} \quad (18)$$

From Theorem 1 it follows that

$$k_N(\omega) < 1 \quad \forall \omega \quad (19)$$

is a necessary and sufficient condition for robust closed-loop stability. Furthermore, the quantity $\gamma = 1/k_N(\omega)$ specifies the amount by which the uncertainty template should be increased or decreased to attain the limiting case of stability.

The SISO critical direction result enables an exact assessment of stability for systems with directional uncertainty templates, without having to circumscribe the templates with larger circular uncertainties. The calculation of the critical perturbation radius $\rho_c(\omega)$ involves determining the intersection of a straight line (the critical direction) with a curve (the boundary of the template $\mathcal{T}(\omega)$). Section 5 shows an example where the critical radius can be calculated analytically.

It is also worth noting that the critical perturbation radius also gives a systematic methodology for determining the uncertainty weights which can be used as one of the inputs for various robust synthesis methods. Of particular interest in this regard are cases where there exists a *critical weighting function*, $W_c(s)$, such that its magnitude satisfies the interpolation condition

$$|W_c(j\omega)| = \rho_c(\omega) \quad \forall \omega \quad (20)$$

When (20) can be exactly satisfied, Theorem 1 leads to the familiar H_∞ condition

$$\left\| \frac{W_c(s)}{1 + g_o(s)} \right\|_\infty < 1 \quad (21)$$

which is of the form (4). If (20) cannot be satisfied exactly but can be approximated using standard frequency-domain regression methods, then the resulting approximate weight can still be used as the basis for practical robust-synthesis design for systems with highly structured templates.

It should be noted that Theorem 1 recovers as a special case the situation where the uncertainties are circular or disk-bounded. The interested reader is referred to Reference 6 for further details on this point.

In the following section we show how the critical direction approach and the Nyquist robust-stability margin may be generalized to the case of uncertain MIMO systems.

4. THE CRITICAL DIRECTION METHOD FOR MIMO SYSTEMS

For MIMO systems we focus attention on the effect of uncertainties on the eigenvalues of the frequency response matrix, along the critical direction. This treatment has several desirable properties. First of all we derive necessary and sufficient robust stability conditions based upon eigenvalue relationships obtained directly from eigenvalue uncertainty templates. Thus, in principle, this method yields a stability margin measure even in cases where singular value conditions fail to give necessary and sufficient stability conditions, provided that a method can be found to define tight inclusion regions for the uncertain eigenvalues. Furthermore, by considering only the subset of the eigenvalue inclusion region that lies along the critical direction, we significantly reduce the computation involved in using the boundary of the eigentemplates (E-contours, see Reference 10) for stability assessment.

Consider the generalized Nyquist plots shown in Figure 2 which (for illustrative purposes only) show two irregularly shaped eigenvalue inclusion regions (templates) about two of the n eigenvalues $\lambda_i(G_o(j\omega_1))$, $i = 1, 2, \dots, n$. In analogy with the SISO case, for each of the n eigenloci we define at each frequency an associated critical line, defined as the directed line originating at the location of nominal eigenvalues $\lambda_i(G_o(j\omega))$ and passing through the critical point $-1 + j0$. Also in analogy with the SISO case discussed in Section 3 we define the following entities:

1. The *critical directions*

$$d_i(j\omega) = - \frac{1 + \lambda_i(G_o(j\omega))}{|1 + \lambda_i(G_o(j\omega))|} \tag{22}$$

which may be interpreted as unit vectors which define the direction of each critical line.

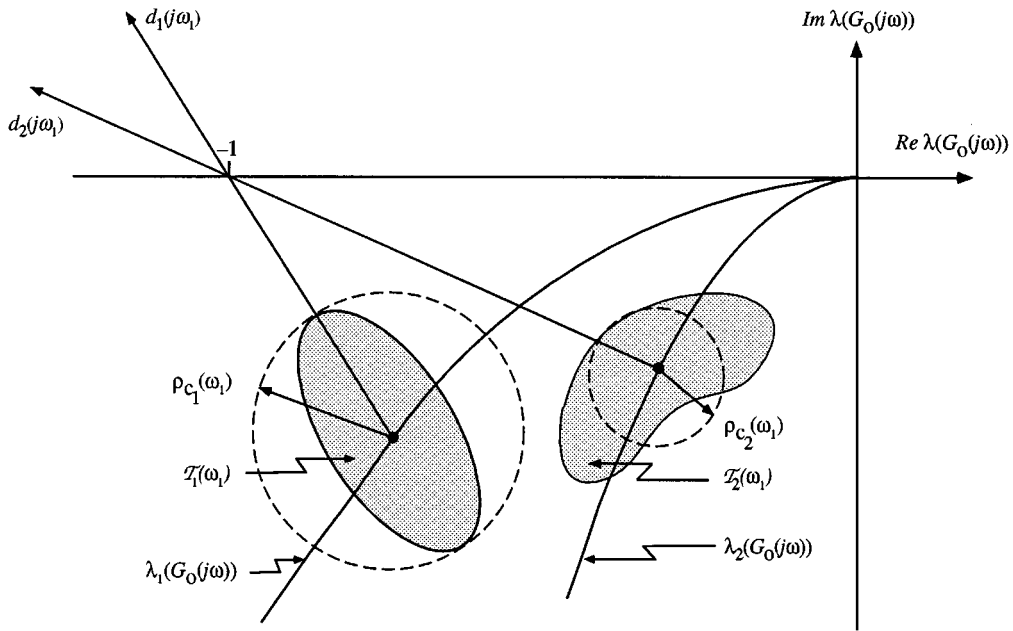


Figure 2. Nyquist plot of the eigenvalues of a 2×2 MIMO system showing two irregularly shaped eigentemplates $\mathcal{T}_1(\omega_1)$ and $\mathcal{T}_2(\omega_1)$ (shaded areas) at frequency $\omega = \omega_1$

2. The *uncertainty eigentemplates*

$$\mathcal{T}_i(\omega) = \{\lambda_i(G(j\omega)) \mid \lambda_i(G(j\omega)) = \lambda_i(G_o(j\omega) + \Delta(j\omega)), \Delta(s) \in \mathcal{D}\} \quad (23)$$

3. The *critical eigentemplates*

$$\mathcal{T}_c(\omega) = \{z \in \mathcal{T}_i(\omega) \mid z = \lambda_i(G_o(j\omega)) + \alpha d_i(j\omega) \text{ for some } \alpha \in \mathcal{R}_+\} \quad (24)$$

4. The *critical perturbation radii*

$$\rho_c(j\omega) = \max_{\alpha_i \in \mathcal{R}_+} \{\alpha_i \mid z = \lambda_i(G_o(j\omega)) + \alpha_i d_i(j\omega) \in \mathcal{T}_c(\omega)\} \quad (25)$$

The entities defined above have identical interpretations to their SISO counterparts defined in Section 2. Hence, their geometrical interpretation can be obtained directly from the SISO Nyquist diagram given in Figure 1, provided that the nominal frequency-response plot $g_o(j\omega)$ is substituted by an eigenvalue plot $\lambda_i(G_o(j\omega))$. Obviously, the tight E-contour templates for unstructured and structured uncertainty descriptions utilized in References 10 and 11 are respectively equivalent to the uncertainty templates $\mathcal{T}_i(\omega)$.

Using these definitions we can now state the following robust stability theorem. As in the SISO case, for simplicity of exposition we assume that the critical uncertainty eigentemplates $\mathcal{T}_c(j\omega)$ are convex sets at all frequencies; however, the templates $\mathcal{T}_i(\omega)$ can be highly nonconvex.

Theorem 2

Let the nominal MIMO system $G_o(s)$ be subject to an uncertainty $\Delta(s) \in \mathcal{D}$. Then the uncertain closed-loop system remains stable under unity feedback if and only if

$$\max_{i=1,2,\dots,n} \frac{\rho_c(\omega)}{|1 + \lambda_i(G_o(j\omega))|} = \frac{\rho_c(\omega)}{|1 + \lambda_c(G_o(j\omega))|} < 1 \quad \forall \omega \quad (26)$$

where $\rho_c(\cdot)$ and $\lambda_c(\cdot)$ are respectively used to denote the critical perturbation radius and the eigenvalue associated with the eigenloci resulting from the maximization over all $i = 1, 2, \dots, n$ in (26).

Proof. Assuming nominal closed-loop stability and that the nominal and perturbed open-loop systems have the same number of open-loop unstable poles, the generalized Nyquist stability criterion guarantees that the uncertain closed-loop system is stable if and only if

$$\lambda_i(G_o(j\omega) + \Delta(j\omega)) \neq -1 \quad \forall i \text{ and } \forall \omega \quad (27)$$

Using the defining equation for the eigenvalues of $G_o(j\omega) + \Delta(j\omega)$, namely

$$\det(G_o(j\omega) - zI + \Delta(j\omega)) = 0 \quad (28)$$

at each frequency ω we can parametrize the uncertain eigenvalues by

$$z = \lambda_i(G_o(j\omega)) + \rho_i e^{j\theta_i}$$

where $\theta_i = \theta_i(\omega)$ varies in the range $0 \leq \theta_i \leq 2\pi$, and for each value of θ_i the scalar $\rho_i = \rho_i(\omega)$ varies in the range $0 \leq \rho_i \leq \bar{\rho}_i$, where $\lambda_i(G_o(j\omega)) + \bar{\rho}_i e^{j\theta_i}$ is an eigenvalue corresponding to the boundary of the eigentemplate $\mathcal{T}(\omega)$. Then condition (27) can be written as

$$1 + \lambda_i(G_o(j\omega)) + \rho_i e^{j\theta_i} \neq 0 \quad \forall i \quad (29)$$

Since the term $1 + \lambda_i(G_o(j\omega))$ is fixed, the only possibility for violating the stability condition (29) is for $\rho_i e^{j\theta_i}$ to be orientated along the critical direction, namely

$$\rho_i e^{j\theta_i} = \alpha_i(\omega) d_i(j\omega)$$

where $0 \leq \alpha_i(\omega) \leq \rho_{c_i}$. Thus for robust stability we have the necessary and sufficient stability condition

$$1 + \lambda_i(G_o(j\omega)) + \alpha_i(\omega) d_i(j\omega) \neq 0 \quad \forall i \text{ and } \forall \omega \quad (30)$$

Invoking definition (22) it follows that a sufficient stability condition is given by

$$\frac{\alpha_i(\omega)}{|1 + \lambda_i(G_o(j\omega))|} < 1, 0 \leq \alpha(\omega) \leq \rho_{c_i}, \quad \forall i \text{ and } \forall \omega \quad (31)$$

Furthermore, since along the critical direction $\alpha_i(\omega) \leq \rho_{c_i}(\omega)$, stability is ensured if

$$\frac{\rho_{c_i}(\omega)}{|1 + \lambda_i(G_o(j\omega))|} < 1 \quad \forall i \text{ and } \forall \omega \quad (32)$$

The proof is completed by taking the maximum over all i and noting that, because of the convexity assumption on the critical templates $\mathcal{T}_{c_i}(\omega)$, conditions (31)–(32) are also *necessary* for stability. \square

In complete parallel with the SISO case

$$k_N(\omega) := \frac{\rho_c(\omega)}{|1 + \lambda_c(G_o(j\omega))|} \quad (33)$$

defines the MIMO *Nyquist robust-stability margin*, and the scalar $\gamma = 1/k_N(\omega)$ again specifies the amount by which the uncertainty template should be stretched or contracted along the critical direction to attain the limiting case of stability. Consequently, from Theorem 2 it follows that a necessary and sufficient condition for robust stability is given by the Nyquist-derived constraint

$$k_N(\omega) < 1 \quad \forall \omega \quad (34)$$

The Nyquist robust stability condition (34) provides an exact answer to the robust stability margin problem, much in the same vein as the margins $k_m(\omega)$ and $\mu(\omega)$.

A most pleasing feature of the new stability measure is that the SISO version (18) and the MIMO version (33) of the Nyquist robust stability margin are formally identical. Much more importantly, the $k_N(\omega)$ formalism adheres to the classical generalization of SISO system properties to the MIMO case via the vehicle of the eigenvalues of the frequency response matrix.

4.1. Relationship between k_N , k_m and μ

In light of Theorem 2 and the remarks above, it is clear that there are strong equivalences between the Nyquist stability margin $k_N(\omega)$ and the margins $k_m(\omega)$ and $\mu(\omega)$. To see this, recall that the definition (8) for k_m is equivalent to finding at each frequency the smallest destabilizing Δ such that

$$1 + \lambda_i(M(j\omega)\Delta(j\omega)) \neq 0 \quad \forall i$$

which is precisely the condition exploited in Theorem 2 to get the necessary and sufficient stability condition

$$1 + \lambda_i(G_o(j\omega)) + \alpha_i(\omega)d_i(j\omega) \neq 0 \quad \forall i \quad (35)$$

For uncertainties which satisfy the condition of Lemma 1, from (11)–(12) it follows that

$$k_N(\omega) < 1 \Leftrightarrow \mu(\omega) < 1 \Leftrightarrow k_m(\omega) > 1 \Leftrightarrow \sup_{\Delta \in \mathcal{D}} \rho(M\Delta) < 1 \quad (36)$$

In fact, even for uncertainties for which Lemma 1 does not hold, it is still true that for all $\Delta(s) \in \mathcal{D}$

$$1 + \lambda(M(j\omega)\Delta(j\omega)) \neq 0 \Leftrightarrow k_N(\omega) < 1$$

The practical utility of Theorems 1 and 2 requires the computation of the critical perturbation radius $\rho_c(\omega)$. Several methods are presently being developed to exploit the critical direction theory in computing $\rho_c(\omega)$ and hence $k_N(\omega)$ for the case of affine and multi-affine parametric uncertainties in SISO and MIMO systems. The results thus far are very encouraging, and have shown significant computational and algorithmic simplifications due to the focus on the critical direction. For the case of structured and unstructured MIMO uncertainties, tight eigenvalue inclusion regions may be obtained using the singular-value based E-contours method,^{10,11} with similarity or non-similarity scaling deployed to remove or reduce conservatism.

5. EXAMPLES

5.1. SISO example

This section presents an example of a SISO system with an uncertainty description motivated from statistical parameter-estimation techniques. The uncertainty considered is an ellipsoidal parameter-space model which, in addition to its intrinsic merits, facilitates a direct analysis in the Nyquist plane which permits an explicit characterization of the Nyquist robust-stability margin $k_N(\omega)$, thus illustrating the application of the critical direction theory proposed in the paper. Furthermore, a discrete-time representation of the model is deliberately chosen to emphasize the fact that the critical direction method and its attendant Nyquist robust-stability margin concept are applicable to both the continuous and discrete domains.

Let us consider a general SISO system model given by the discrete-time transfer function

$$H(z) = \sum_{k=1}^q h_k z^{-k} \quad (37)$$

with q uncertain parameters defined by $\mathbf{h} = [h_1, h_2, \dots, h_q]^T$. The nominal system, denoted $H_o(z)$, is obtained when \mathbf{h} assumes the nominal values $\mathbf{h}_0 = [h_1^0, h_2^0, \dots, h_q^0]^T$, so that the real system is modelled as $H(z) = H_o(z) + \delta H(z)$.

Let $\mathbf{h} \in \mathcal{R}^q$ be such that $\mathbf{h} = \mathbf{h}^0 + \delta\mathbf{h}$. Then we define the uncertain ellipsoidal parametric uncertainty description

$$\delta\mathbf{h} = \mathbf{h} - \mathbf{h}^0, \quad \delta\mathbf{h} \in \mathcal{D}_h \quad (38)$$

$$\mathcal{D}_h := \{ \delta\mathbf{h} \in \mathcal{R}^q \mid \delta\mathbf{h}^T \mathbf{Q}_h^{-1} \delta\mathbf{h} \leq 1; \quad \mathbf{Q}_h = \mathbf{Q}_h^T > 0 \} \quad (39)$$

We argue that, in addition to the remarkable mathematical tractability which we shall show later, the ellipsoidal parametric description of uncertainties is quite natural in many applications and

offers, in contrast to hyperrectangular descriptions, the further advantage of allowing the dependence among various system parameters to be taken into explicit account. Ellipsoidal models often arise quite naturally, as for example whenever linear regression or least-squares analysis are used in model estimation (see for example References 5, 12 and 13).

Clearly the nominal parameter vector \mathbf{h}_0 defines the nominal transfer function $H_0(z) = \sum_{k=1}^q h_k^0 z^{-k}$. Consider now the frequency response $H(e^{j\omega})$ where $H(z) = \sum_{k=1}^q h_k z^{-k}$ contains parameters $\mathbf{h} = [h_1, h_2, \dots, h_q]^T$ belonging to the parameter ellipsoid. Under these conditions the following lemma shows that the parameter space ellipsoid maps precisely to ellipses at all frequencies except for $\omega = 0$ and $\omega = \pi$. Let $\mathcal{R}(z)$ and $\mathcal{I}(z)$ represent respectively the real and imaginary components of a complex number z .

Lemma 2

The parameter space ellipsoid defined by (39) maps to the elliptical uncertainty template

$$\mathcal{F}(\omega) = \{z(\omega) = x_1(\omega) + jx_2(\omega) \mid (\mathbf{x}(\omega) - \mathbf{x}_0(\omega))^T \mathbf{Q}_\omega^{-1} (\mathbf{x}(\omega) - \mathbf{x}_0(\omega)) \leq 1, \omega \in (0, \pi)\} \quad (40)$$

where $\mathbf{x}(\omega) := [x_1(\omega) \ x_2(\omega)]^T \in \mathcal{R}^2$, $\mathbf{x}_0(\omega) := [x_1^0(\omega) \ x_2^0(\omega)]^T \in \mathcal{R}^2$, such that

$$\begin{aligned} \mathbf{x}(\omega) &= \begin{bmatrix} \mathcal{R}H(e^{j\omega}) \\ \mathcal{I}H(e^{j\omega}) \end{bmatrix} = \mathbf{V}(\omega)\mathbf{h} \\ \mathbf{x}_0(\omega) &= \begin{bmatrix} \mathcal{R}H_0(e^{j\omega}) \\ \mathcal{I}H_0(e^{j\omega}) \end{bmatrix} = \mathbf{V}(\omega)\mathbf{h}_0 \\ \mathbf{Q}_\omega &:= \mathbf{V}(\omega)\mathbf{Q}_h\mathbf{V}(\omega)^T \in \mathcal{R}^{2 \times 2} \\ \mathbf{V}(\omega) &= \begin{bmatrix} \cos\omega & \cos 2\omega & \dots & \cos q\omega \\ \sin\omega & \sin 2\omega & \dots & \sin q\omega \end{bmatrix} \end{aligned}$$

At $\omega = 0$ and $\omega = \pi$, matrix $\mathbf{V}(\omega)$ is singular and the uncertainty template is entirely real and given by

$$\mathcal{F}(\omega) = \{z(\omega) = x_1(\omega) \mid x_2(\omega) = 0, |x_1(\omega) - x_1^0(\omega)| \leq \sqrt{\mathbf{v}_R^T(\omega)\mathbf{Q}_h\mathbf{v}_R(\omega)}, \omega = 0, \pi\} \quad (41)$$

where $\mathbf{v}_R(\omega) = [\cos\omega \ \cos 2\omega \ \dots \ \cos q\omega]^T$.

Proof. The proof makes use of a result given in Reference 5. Details are omitted. \square

Our intention is to characterize the Nyquist robust-stability margin for the unity feedback system comprised of the nominal discrete-time system and its associated parametric uncertainty ellipsoid (39). Note that the robust stability of the closed-loop system cannot be analysed using the l^2 results in Reference 14 because their method is applicable only to continuous-time systems and to ellipsoidal uncertainty descriptions where the matrix \mathbf{Q}_h is diagonal (i.e., the principal directions are aligned with the co-ordinate axes). In fact, none of the conventional methods for analysing robust stability of SISO systems appears capable of treating in a systematic fashion the case of ellipsoidal uncertainties \mathcal{D}_h considered in this example.

Theorem 3

Under the assumptions of nominal closed-loop stability and that the nominal and perturbed systems share the same number of open-loop unstable poles, the unity negative feedback system

with open-loop transfer function (37) and uncertain parameters (39) is stable for all $\delta \mathbf{h} \in \mathcal{D}_{\mathbf{h}}$ if and only if

$$k_N(\omega) < 1 \quad \forall \omega \quad (42)$$

where

$$k_N(\omega) = \frac{1}{\sqrt{\mathbf{d}_c^T(\omega) \mathbf{Q}_\omega^{-1} \mathbf{d}_c(\omega)}}, \quad \omega \in (0, \pi) \quad (43)$$

$$k_N(\omega) = \sqrt{\frac{\mathbf{v}_R^T(\omega) \mathbf{Q}_h \mathbf{v}_R(\omega)}{\mathbf{d}_c^T(\omega) \mathbf{d}_c(\omega)}}, \quad \omega = 0, \pi \quad (44)$$

and

$$\mathbf{d}_c(\omega) := \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \mathbf{V}(\omega) \mathbf{h}_0 \quad (45)$$

Proof. Inequality (42) is a direct result of applying the necessary and sufficient condition (17) of Theorem 1 in the Nyquist robustness margin form (19). It then suffices to prove (43) and (44). First consider the case where $\omega \in (0, \pi)$. Using two-dimensional vector analysis and the definitions given in Section 3, it is clear that the components of $\mathbf{d}_c(\omega)$ given by (45) are respectively the unnormalized real and imaginary parts of the critical direction $d(j\omega)$ as defined in (22). Now from Lemma 2, at each ω the frequency-response $H(e^{j\omega})$ is located inside an elliptical template with centre at the point $H_0(e^{j\omega})$. Clearly, the elliptical template is a convex set because it is continuous along any ray. A fundamental result from two-dimensional co-ordinate geometry gives the length of the line joining the centre of an ellipse to the point of intersection along the vector $\mathbf{d}_c(\omega)$ as

$$\rho_c(\omega) = \sqrt{\frac{\mathbf{d}_c^T(\omega) \mathbf{d}_c(\omega)}{\mathbf{d}_c^T(\omega) \mathbf{Q}_\omega^{-1} \mathbf{d}_c(\omega)}} \quad (46)$$

Recognizing that $|1 + H_0(e^{j\omega})| = \sqrt{\mathbf{d}_c^T(\omega) \mathbf{d}_c(\omega)}$ it readily follows that $k_N(\omega) = \rho_c(\omega)/|1 + H_0(e^{j\omega})|$ is of the form (43). Analogously, the proof for the singular cases $\omega = 0, \pi$ is derived noting that from Lemma 2 it follows that at these frequencies $\rho_c(\omega) = \sqrt{\mathbf{v}_R^T(\omega) \mathbf{Q}_h \mathbf{v}_R(\omega)}$, from which (44) is readily established. \square

In summary, for the case of ellipsoidal parametric uncertainties, the critical direction theory permits the exact characterization of the Nyquist robust-stability margin, and the derivation of the exact necessary and sufficient condition (42) for robust stability.

5.2. MIMO example

Consider the MIMO nominal transfer function

$$G_\theta(s) = 0.5 \begin{bmatrix} \frac{30}{(s+1)(s+2)(s+3)} & \frac{-3}{s+4} \\ \frac{7}{s+5} & \frac{10}{s+1} \end{bmatrix} \quad (47)$$

and the associated element-by-element elliptical uncertainty description

$$\mathcal{D}_E := \{\Delta(s) \mid \Delta_{ik}(j\omega) \in \mathcal{E}_{ik}(j\omega), i, k = 1, 2\} \quad (48)$$

where the boundary $\partial\mathcal{E}_{ik}(j\omega)$ of each elliptical domain $\mathcal{E}_{ik}(j\omega)$ is given by the map

$$\partial\mathcal{E}_{ik}(j\omega) = 2A_{ik}e^{j\phi_{ik}} \cos\theta_{ik} + j2B_{ik}e^{j\phi_{ik}} \sin\theta_{ik}, i, k = 1, 2 \quad (49)$$

and where the coefficients A_{ik} and B_{ik} are the elements of the frequency-dependent matrices $A(\omega)$ and $B(\omega)$, and the major-axis orientation parameters ϕ_{ik} are the elements of the matrix $\Phi(\omega)$. The objective is to illustrate the calculation of the Nyquist robust stability margin $k_N^{\mathcal{D}_E}$ at the frequency $\omega_1 = 1.21$, where

$$A(\omega_1) = \begin{bmatrix} 0.1264 & 0.0359 \\ 0.0680 & 0.3185 \end{bmatrix}, B(\omega_1) = \begin{bmatrix} 0.0246 & 0.0278 \\ 0.0537 & 0.2707 \end{bmatrix}, \Phi(\omega_1) = \begin{bmatrix} 2.1799 & 4.5759 \\ 1.4906 & 3.8865 \end{bmatrix} \quad (50)$$

Note that formulations based on the multivariable stability margin $k_m(\omega)$ or the structured singular value $\mu(\omega)$ cannot currently offer an obvious approach for calculating the stability margins for the uncertainty description \mathcal{D}_E .

At each frequency of interest we propose the following calculation procedure. First, the nominal eigenvalues $\lambda_i(G_o(j\omega))$ and the critical directions $d_i(\omega)$, $i = 1, 2$, are computed.

Second, for each eigenvalue, z satisfying

$$\bar{\sigma}(DE_2(G_o - (\lambda_i + \rho_i d_i)^{-1} E_1 P D^{-1})) = 1$$

is found. In the above expression, D is a diagonal similarity scaling matrix, P is the diagonal matrix where each diagonal element is the radius of the circle circumscribing the elliptical uncertainties. E_1 and E_2 are derived from the diagonalization of the uncertainty⁴ and λ_i represents the eigenvalues of G_o . This yields the upper bound for $\lambda_i(G_o + \Delta)$ in the critical direction. Then, an optimization is carried out over all θ_{ik} minimizing the function

$$f = |z_i^0 - \lambda_i(G_o + \Delta)|^2 + \left| d_i - \frac{1 + \lambda_i(G_o + \Delta)}{|1 + \lambda_i(G_o + \Delta)|} \right|^2$$

This minimization yields the eigenvalue of $(G_o + \Delta)$ in the critical direction which is closest to the upper bound z_i^0 . The basic principle is similar to that described in Reference 15.

Finally, the critical radii are obtained as $\rho_{c_i} = |z_i(j\omega) - \lambda_i(G_o(j\omega))|$. Finally, the stability margin is directly calculated from the definitions (26) and (33).

At the selected frequency $\omega_1 = 1.21$, the nominal eigenvalues for the system considered in this example are $\lambda_1(G_o(j\omega_1)) = -0.2005 - j1.2313$ and $\lambda_2(G_o(j\omega_1)) = 1.9332 - j2.4525$, and they have the associated critical directions $d_1(j\omega_1) = -0.5446 + j0.8387$ and $d_2(j\omega_1) = -0.7672 + j0.6414$. Then using the structured E -contour method for the fixed critical directions $d_1(j\omega_1)$ and $d_2(j\omega_1)$ yields the critical radii $\rho_{c_1}(\omega) = 0.3140$ and $\rho_{c_2} = 0.6445$. Finally, from (26) and (33) it is found that

$$k_N^{\mathcal{D}_E}(\omega_1) = \max \left\{ \frac{\rho_{c_1}(\omega_1)}{|1 + \lambda_1(G_o(j\omega_1))|}, \frac{\rho_{c_2}(\omega_1)}{|1 + \lambda_2(G_o(j\omega_1))|} \right\} = \max \{0.2139, 0.1686\} \quad (51)$$

Hence, the Nyquist robust stability margin is $k_N^{\mathcal{D}_E}(\omega_1) = 0.21 < 1$, and it is concluded that the system satisfies the necessary and sufficient condition for robust stability at this particular frequency.

Figure 3 shows a Nyquist diagram with the eigenplots for the nominal system considered in this example. The eigentemplates $\mathcal{T}_1(\omega_1)$ and $\mathcal{T}_2(\omega_1)$ are not shown in the figure because, for the purpose of determining the Nyquist robust-stability margin, it is not necessary to calculate the entire templates. However, for reference, the figure shows the optimal D -scaling bounds obtained using the scalings given in Reference 4. The actual eigentemplates are bounded by the optimal D -scaling contours. The points denoted $z_1(j\omega_1)$ and $z_2(j\omega_1)$ are on the boundary of the critical templates $\mathcal{T}_{c_1}(\omega_1)$ and $\mathcal{T}_{c_2}(\omega_1)$, respectively, and were obtained using the E -contour method. Note that for this example the D -scaling bounds provide a good estimate for the boundary point $z_1(j\omega_1)$, but yield a poor estimate for $z_2(j\omega_1)$. Note also that the critical eigentemplates $\mathcal{T}_{c_1}(\omega_1)$ and $\mathcal{T}_{c_2}(\omega_1)$ can be readily identified from the figure as the straight-line segments joining each nominal eigenvalue $\lambda_i(G_o(j\omega_1))$ with its corresponding boundary point $z_i(j\omega_1)$.

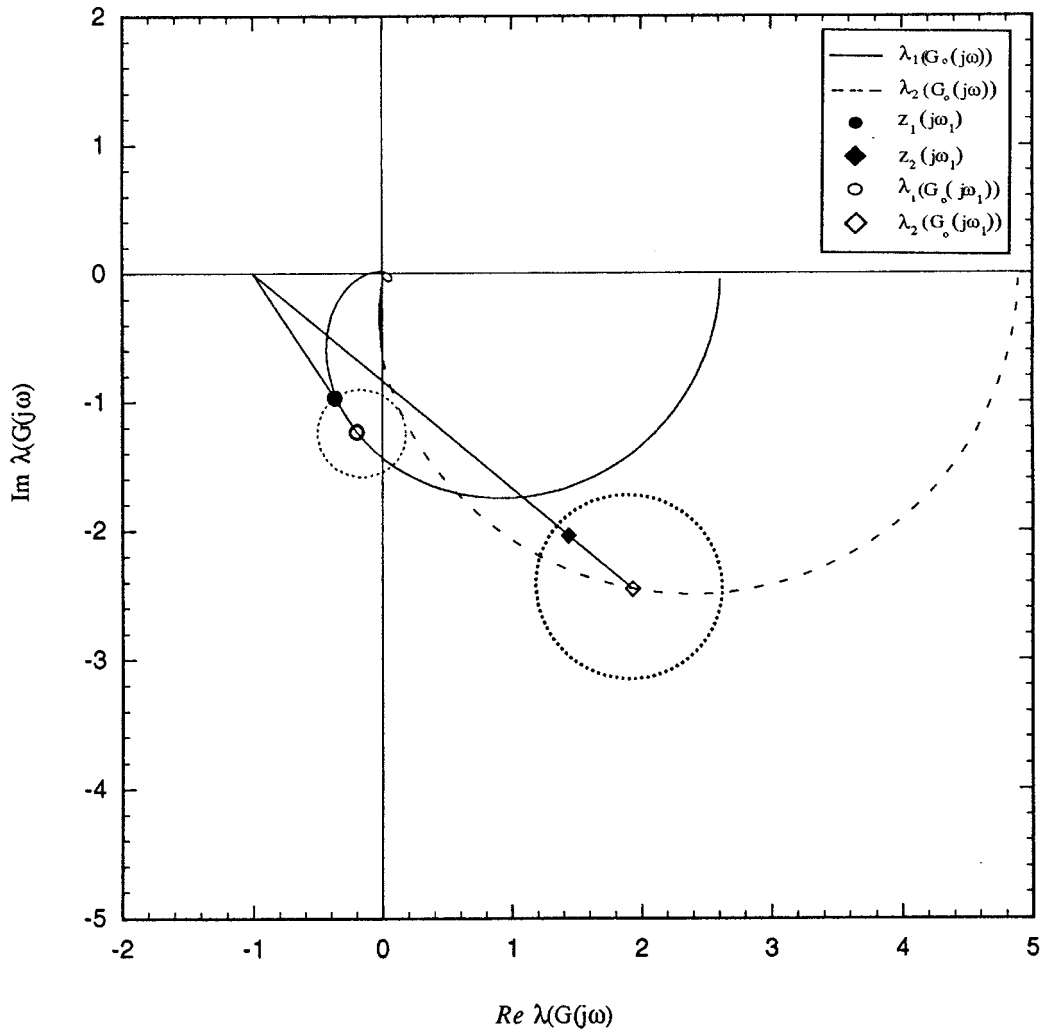


Figure 3. Nyquist plot of the MIMO system of Example 2. The contours represent the optimal D -scaling bounds for the eigentemplates $\mathcal{T}_1(\omega_1)$ and $\mathcal{T}_2(\omega_1)$ at the frequency $\omega_1 = 1.21$. The points $z_1(j\omega_1)$ and $z_2(j\omega_1)$ lie on the intersection of the boundary of their respective eigentemplates with the critical direction

6. CONCLUSIONS

In this paper we have proposed the Nyquist Stability Margin, $k_N(\omega)$ as a new metric for robustness analysis of SISO and MIMO systems. The definition of new stability margin is based on the ‘critical direction theory’ which provides a single framework for robustness analysis for SISO and MIMO systems. The analysis methodology makes direct use of the generalized Nyquist diagram, and in contrast to the prevalent approaches which emphasize singular-value perturbations, it focuses attention on eigenvalue perturbations.

The main advantage of the critical direction theory is that it provides necessary and sufficient conditions for robust stability in the presence of highly structured uncertainties with phase and directionality constraints. Other approaches to these problems either do not have the inherent capability to deal with these structural details, or the directionality and phase constraints are deliberately ignored, giving rise in either case to sufficient-only conditions such as those associated with singular-value theory. On the other hand, the new method explicitly exploits the detailed directionality and phase constraints of the uncertainties as these are manifested in the frequency domain uncertainty templates. Thus, the new method is applicable to a number of uncertainty descriptions for which other methods fail, such as the case of element-by-element ellipsoidal uncertainties in the transfer-function matrix, and other uncertainty descriptions with highly directional frequency-domain templates.

The new critical direction technique opens up new avenues for robustness analysis and could lead to novel approaches for robust control synthesis. There is significant promise for fruitful new results in this area where the computational efforts are concentrated on a single and well-defined frequency-dependent directed line.

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