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A New Method for Computing Robustness Margins for Real Parametric Uncertainties

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Abstract

A new method is proposed for computing the robust stability margin for a single-input single-output systems subject to real interval uncertainties that appear multiaffinely in the system transfer function. The approach utilizes the recently developed critical direction theory to study the robust stability problem directly in the Nyquist plane, and permits the calculation of the robust stability margin using a bisection search. A further benefit of the method is that a reduced range of frequencies yielding the robustness margin can be readily identified. An algorithm is presented which utilizes both of these properties, resulting in a significant reduction in computational burden.

1. Introduction

Maintaining system stability in the face of uncertainty is a fundamental issue in control design. The problem of computing robust stability margins for systems affected by uncertainty is often considered in the so called M-Δ framework, where the margin is interpreted as "smallest" size of the uncertainty Δ that leads to instability. Several different norms may be used to measure the size of Δ [1], but the most common is the H_∞ norm which leads to the definition of μ, the structured singular value [2], or its inverse, k_m, the multivariable stability margin [3].

For the particular case of affine or multiaffine real parametric uncertainties, the domain-splitting technique of de Gaston and Safonov [5] can be used to compute the stability margin. In this paper we present an alternative method for computing the robust stability margin directly in the Nyquist plane. The technique calls for the direct mapping of the parameter-space uncertainty into the Nyquist plane, hence avoiding the use of an augmented M-Δ formulation.

The foundation for the proposed method is the recently developed critical-direction theory [6] [7]. An algorithm based on this theory is proposed to identify both a reduced range of frequency values and a reduced range of uncertainty scale factors which must contain the stability margin. A bisection search on the scaling factors is then performed to find the stability margin.

2. Preliminaries

Consider the linear uncertain system

$$g(s,\boldsymbol{q}) = \frac{N(s,\boldsymbol{q})}{D(s,\boldsymbol{q})} \tag{1}$$

where $\boldsymbol{q} = [q_1, q_2, \ldots, q_n]^T \in Q$ is a vector of uncertain parameters, and Q is a rectangular polytope, whose center is assumed to be at $\boldsymbol{q} = \boldsymbol{0}$, *i.e.*,

$$Q = \left\{ \boldsymbol{q} \in \Re^n \;\middle|\; |q_i| \le \beta_i,\ \beta_i > 0, i = 1,2,\ldots,n \right\}$$

The notation

$$\alpha Q = \left\{ \boldsymbol{q} \in \Re^n \;\middle|\; |q_i| \le \alpha\beta_i, i = 1,2,\ldots,n \right\}$$

represents a scaled uncertainty polytope featuring a positive factor α. The coefficients of $N(s,\boldsymbol{q})$ and $D(s,\boldsymbol{q})$ are assumed to depend either affinely or multiaffinely on $\boldsymbol{q}$. The nominal system is $g^0(s) = g(s,\boldsymbol{0})$, and it is assumed that the nominal system is stable under a unity feedback configuration.

A frequency-dependent *parametric* robust stability margin for the unity-feedback closed loop involving system (1) is defined as

$$\alpha(\omega) = \min_{\alpha \in \Re^+} \left\{ \alpha \;\middle|\; 1 + g(j\omega,\boldsymbol{q}) = 0,\ \boldsymbol{q} \in \alpha Q \right\} \tag{2}$$

Furthermore, the overall parametric robust stability margin is given by

$$\alpha^* := \min_{\omega} \alpha(\omega) \tag{3}$$

In the M-Δ framework, the multivariable stability margin is defined by de Gaston and Safonov [5] as $k_m^* = \min_{\omega} k_m(\omega)$, where

$$k_m(\omega) = \min_{k \in \Re^+} \left\{ k \;\middle|\; \det[\mathrm{I} - kM(j\omega)\Delta(j\omega)] = 0 \right\} \tag{4}$$

The case where $\Delta(s)$ contains only real scalars leads to the equivalence $k_m(\omega) = \alpha(\omega)$, provided the assumption that the uncertainty structure is multiaffine is satisfied. However, the method presented in [5] requires an exhaustive numerical search in the frequency domain (by partitioning the frequency range into small intervals) plus an exhaustive search in the parameter domain (by repeatedly subdividing the polytope into small domains). The use of the critical direction theory allows significant reductions in the size of the frequency and parameter domains that must be investigated numerically.

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3. Critical Direction Theory

The critical-direction theory [6] [7] is a general frequency domain technique for quantifying the stability robustness of a feedback loop subject to uncertainty. The approach is characterized by the following entities:

i. The *critical direction,*

$$d(j\omega) = -\frac{1+g^0(j\omega)}{\left|1+g^0(j\omega)\right|}$$

is defined as the direction of the oriented line originating at the nominal point $g^0(j\omega)$ and passing through the point $-1+j0$.

ii. The *critical uncertainty template*

$$\mathcal{T}_c(\omega,Q) = \{\ g(j\omega,\boldsymbol{q})\ \mid g(j\omega,\boldsymbol{q}) = g^o(j\omega)+\gamma \boldsymbol{d}(j\omega)$$

$$\text{for all } \gamma \in \Re^+ \text{ and all } \boldsymbol{q} \in Q\ \}$$

iii. The *critical perturbation radius*

$$\rho_c(\omega,Q) = \max_{\gamma\in\Re^+}\left\{\gamma\ \mid g^o(j\omega)+\gamma \boldsymbol{d}(j\omega) \in \mathcal{T}_c(\omega,Q)\right\}$$

iv. The *Nyquist robust stability margin*

$$k_N(\omega,Q) := \frac{\rho_c(\omega,Q)}{\left|1+g^o(j\omega)\right|} \tag{5}$$

Theorem 1. *Consider the uncertain system (1) and suppose that the nominal system is stable under unity negative feedback, and that $g(s,q)$ and $g^o(s)$ have the same number of open-loop unstable poles for all $\boldsymbol{q} \in Q$. Then the uncertain system is stable under unity negative feedback if and only if*

$$k_N(\omega,Q) < 1 \ \ \forall\omega \tag{6}$$

A complete proof of the necessity and sufficiency of the result is given in [6] and [8]. The theorem holds provided $\mathcal{T}_c(\omega,Q)$ is a convex set, *i.e.*, whenever $\mathcal{T}_c(\omega,Q)$ is represented by a continuous straight-line segment in the Nyquist plane. Note that the entire value set may be nonconvex; only the critical subset is assumed convex.

4. Computing Stability Margins Using the Critical-Direction Theory

The Nyquist robust stability margin k_N gives an exact measure of the closeness to instability of a particular value set. Consider the scaling of the parametric uncertainty set Q to obtain αQ. Then all values of α that violate the inequality $k_N(\omega,\alpha Q) < 1$ lead to instability of the closed loop. Clearly, the scaling factor $\alpha(\omega)$ that yields the equality

$$k_N(\omega,\alpha(\omega)Q) = 1$$

at a particular frequency ω is precisely the parametric stability margin defined in (2). This follows from the fact that this critical stability condition occurs when the condition $1+g(j\omega,\boldsymbol{q})=0$ is satisfied for some $\boldsymbol{q} \in \alpha Q$.

4.1 Computation of the Critical Perturbation Radius

To compute the robustness margin one must be able to calculate $k_N(\omega,Q)$ from (5), which in turn requires computation of the critical perturbation radius $\rho_c(\omega,Q)$. In turn, the boundary of the value set along the critical direction must be found in order to calculate $\rho_c(\omega,Q)$.

Recent results by Fu [9] and Fu *et al.* [10] show that the boundaries of the value set in the Nyquist plane are images of points located on the edges and on a set of well-defined interior line segments of the polytope Q. We make use of this result to propose a systematic procedure to determine which edges and interior line segments contain points that intersect with the critical direction when mapped to the Nyquist plane.

Consider the definitions $x_0 = \mathrm{Re}\{g^0(j\omega)\}$, $y_0 = \mathrm{Im}\{g^0(j\omega)\}$, $x = \mathrm{Re}\{g(j\omega,\boldsymbol{q})\}$, $y = \mathrm{Im}\{g(j\omega,\boldsymbol{q})\}$, and $y = \mathrm{Im}\{g(j\omega,\boldsymbol{q})\}$, and the test function $L(\boldsymbol{q}) = y(x_0+1) - y_0(x+1)$. At a fixed frequency ω, any uncertainty vector $\boldsymbol{q}$ that maps to a Nyquist-plane point $g(j\omega,\boldsymbol{q})$ lying on the critical direction satisfies the equation $L(\mathbf{q})=0$. This permits the execution of an exhaustive numerical search over all the relevant segments of the uncertainty polytope. Any uncertainty vector $\bar{q}_i$ that satisfies $|L(\bar{q}_i)|<\delta$, for some user-specified tolerance δ, is considered to produce an image $g(j\omega,\bar{q}_i)$ that lies along the critical direction. This process is repeated until all edges that could potentially contribute to generate the boundary of the value set are checked. The process yields a series of parameter vectors $\{\bar{q}_i\}$ whose Nyquist-plane images lie on the critical direction. Finally, the image point that lies furthest from $g^o(j\omega)$ along the critical direction defines $\rho_c(\omega,Q)$. This information allows the calculation of $k_N(\omega,\alpha Q)$ using (5).

4.2 Proposed Robustness Margin Algorithm

The process for calculating $\rho_c(\omega,\alpha Q)$ for any given magnification factor α given in Section 4.1 can be used for calculating α^* by (*i*) carrying out the minimization (2) via an exhaustive numerical search over $\alpha \in \Re^+$, and (*ii*) carrying out the minimization (3) via another numerical search over the frequency domain. Unfortunately, this approach involves significant numerical effort.

The critical direction theory can be used to increase the accuracy and reduce the computational effort involved in the minimization (2) by introducing a bisection-search algorithm, and to also reduce the computational effort involved in the minimization (3) by restricting the frequency search to selected relevant intervals. The following lemma is used to justify the feasibility of the bisection search.

Lemma 1. *Consider the interval plant (1) along with the scaling factor* $\alpha \in \Re^+$. *Then the critical perturbation radius* $\rho_c(\omega,\alpha Q)$ *is a non-decreasing function of* α.

Proof. Let Q be the uncertainty polytope, and consider two positive scaling factors α_1 and α_2, such that $\alpha_2 > \alpha_1$. Let $\mathcal{T}_i$ denote the value set mapped by $\alpha_i Q$ at a particular frequency, for i=1,2. The value set $\mathcal{T}_2$ that is generated by $\alpha_2 Q$ must contain completely the original value set $\mathcal{T}_1$ because $\alpha_1 Q \subset \alpha_2 Q$. This implies $\mathcal{T}_1 \subset \mathcal{T}_2$. The critical perturbation radius $\rho_c(\omega,\alpha_2 Q)$ is the distance from the nominal point (which lies inside $\mathcal{T}_2$) to the edge point of $\mathcal{T}_2$ that lies on the critical direction. It follows that $\rho_c(\omega,\alpha_2 Q) \geq \rho_c(\omega,\alpha_1 Q)$, implying that $\rho_c(\omega,\alpha Q)$ is a non-decreasing function of α. ∇

The proposed method is carried out by the following algorithm consisting of six steps.

(S1) Set the iteration number n=1, an initial (low) value for α_l and termination tolerance parameters ε>0, and δ>0. Establish an initial set of frequency points to be considered $\mathcal{F} = \{ \omega_k , k=1,\ldots,m\}$.

(S2) Compute $\left|1+g^o(j\omega_k)\right|$ for all points $\omega_k \in \mathcal{F}$.

(S3) Compute $\rho_c(\omega_k,\alpha_n Q)$ to a tolerance δ as shown in Section 4.1 and calculate $k_N(\omega_k,\alpha_n Q)$ for all $\omega_k \in \mathcal{F}$.

  (3a) If $k_N(\omega_k,\alpha_n Q) < 1-\varepsilon$ for all $\omega_k \in \mathcal{F}$, set $\alpha_{n+1} = 2\alpha_n$ and repeat S3.

  (3b) If $k_N(\omega_k,\alpha_n Q) > 1+\varepsilon$ at any frequency $\omega_k \in \mathcal{F}$, set $\alpha_{up} = \alpha_n$, $\alpha_{lo} = \alpha_{n-1}$, and go to S4.

  (3c) If $1-\varepsilon < k_N(\omega_i,\alpha_n Q) < 1+\varepsilon$ for some $\omega_i \in \mathcal{F}$, and $k_N(\omega_k,\alpha_n Q) < 1-\varepsilon$ for all $\omega_k \in \mathcal{F}$, $\omega_k \neq \omega_i$, then stop and set $\alpha^* = \alpha_n$.

(S4) Reduce the frequency range to $\mathcal{F} = \{ \omega_k \,|\, \omega_{lo} \leq \omega_k \leq \omega_{up}\}$ where, for all $\omega_{lo} \leq \omega_k \leq \omega_{up}$, $k_N(\omega_k,\bar{\alpha} Q) > 1+\varepsilon$. (Note that there may be more than one range).

(S5) Bisect the scale by setting $\bar{\alpha} = (\alpha_{up} + \alpha_{lo})/2$.

(S6) Compute the critical perturbation radius $\rho_c(\omega_k,\bar{\alpha} Q)$ and calculate $k_N(\omega_k,\bar{\alpha} Q)$ for all $\omega_k \in \mathcal{F}$.

  (6a) If $k_N(\omega_k,\bar{\alpha} Q) < 1-\varepsilon \ \forall \omega_k$, set $\alpha_{lo} = \bar{\alpha}$ and go to S5.

  (6b) If $k_N(\omega_k,\bar{\alpha} Q) > 1+\varepsilon$ at any $\omega_k \in \mathcal{F}$, set $\alpha_{up} = \bar{\alpha}$, $\alpha_n = \bar{\alpha}$, and go to S4 to further reduce the frequency range.

  (6c) If $k_N(\omega_i,\bar{\alpha} Q) > 1-\varepsilon$ for some $\omega_i \in \mathcal{F}$, and $k_N(\omega_k,\bar{\alpha} Q) < 1-\varepsilon$ for all $\omega_k \in \mathcal{F}$, $\omega_k \neq \omega_i$, then stop and set $\alpha^* = \bar{\alpha}$.

5. Conclusions

A method for calculating robust stability margins for multiaffine SISO uncertainties shows the benefits of reduced computational offers due to the identification of restricted relevant ranges of frequency and by the introduction of a bisection search method that is more precise and more efficient than exhaustive numerical searches.

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